Three tree problems

Benoît Corsini

NUS Math Workshop in Ha Noi

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Quicksort

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4	5	2	7	1	3	6

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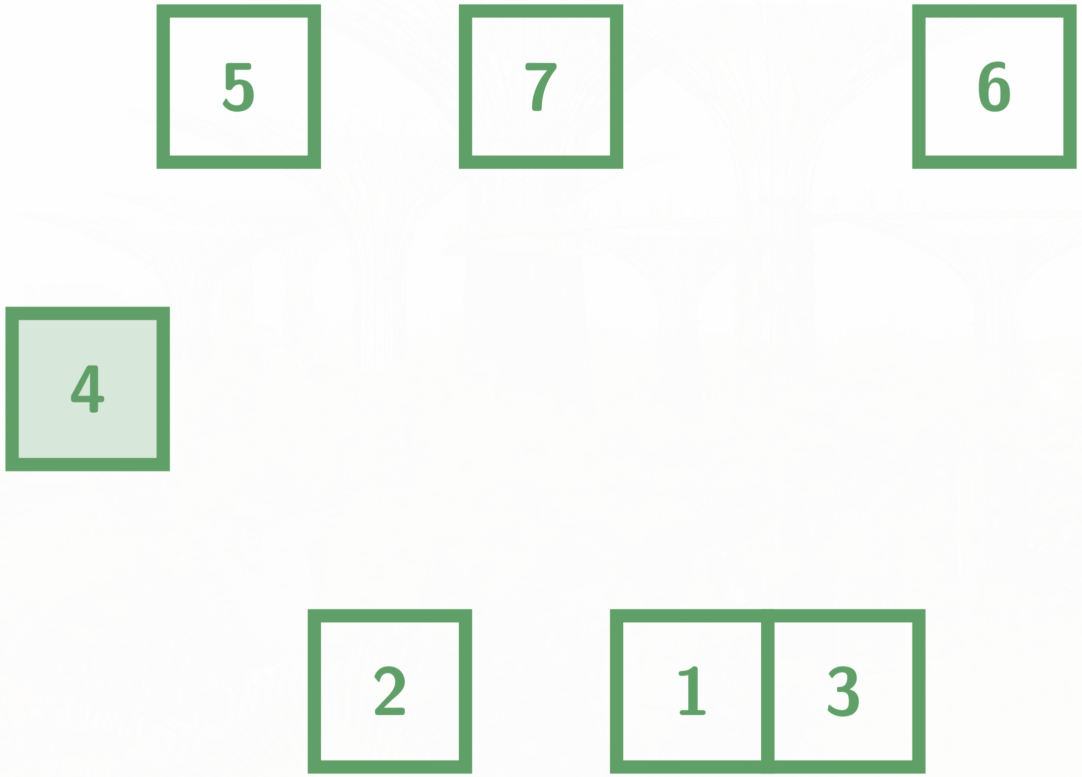

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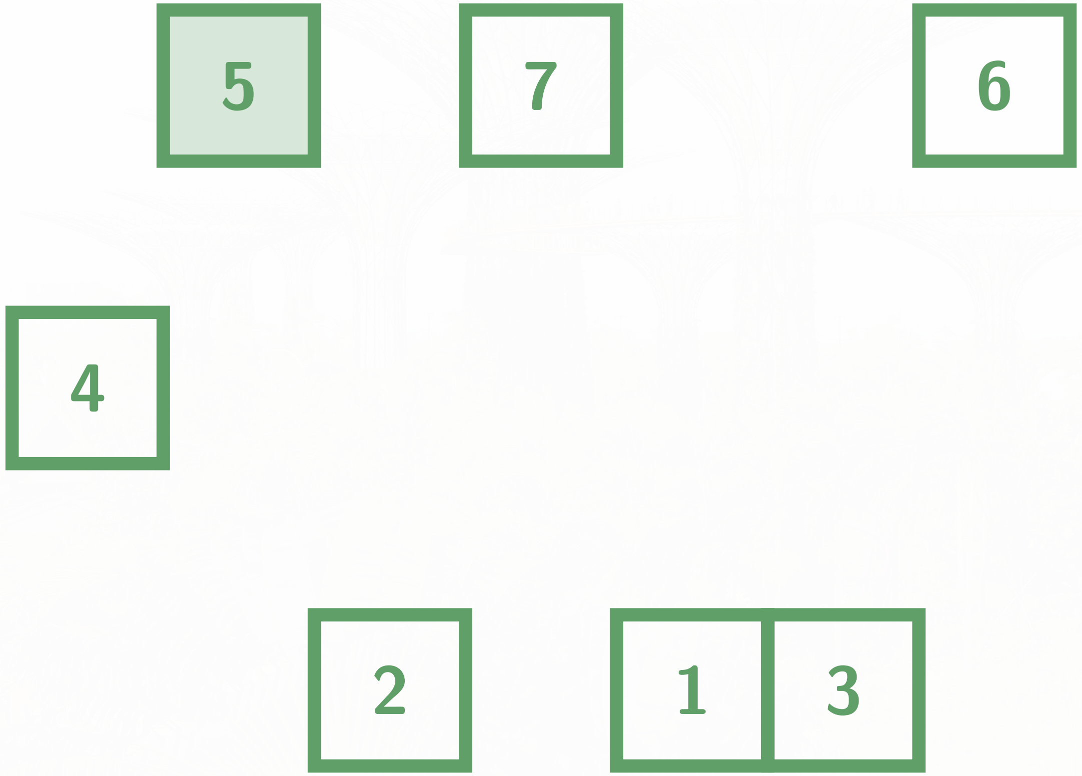

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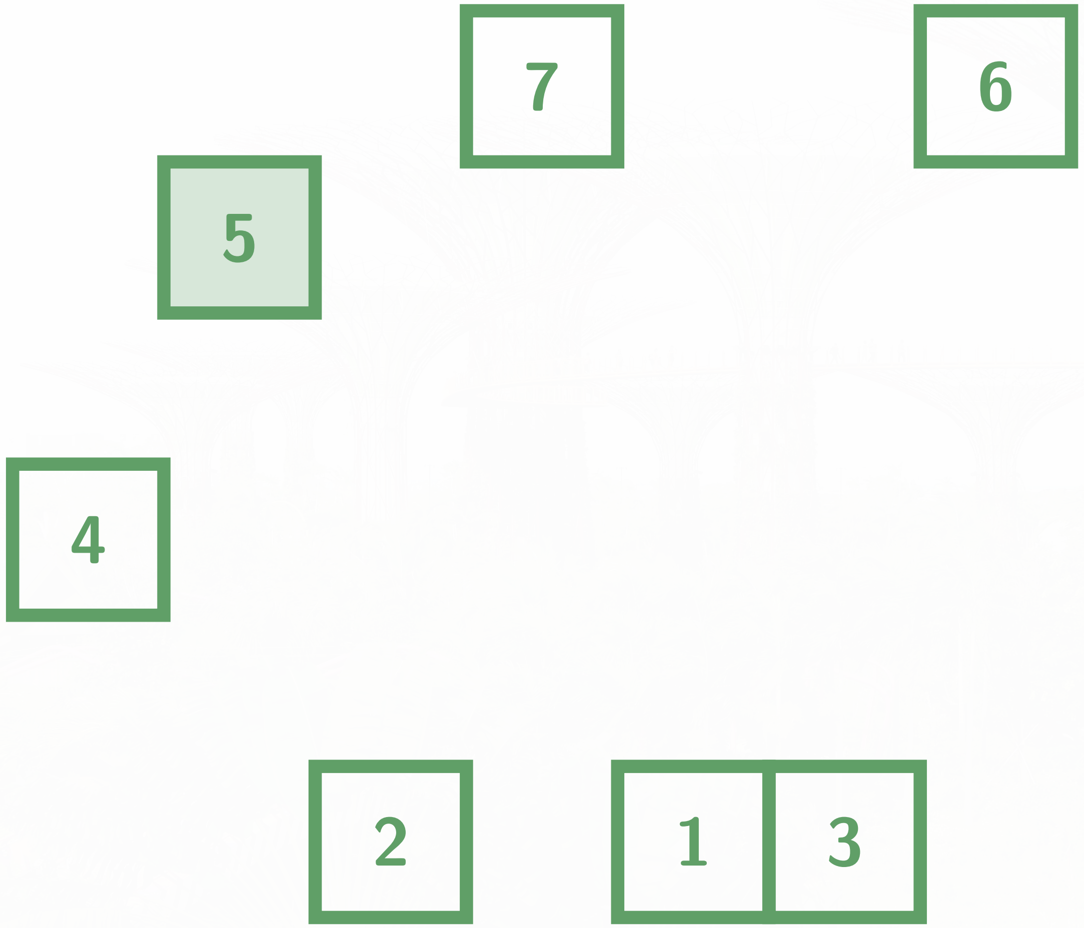

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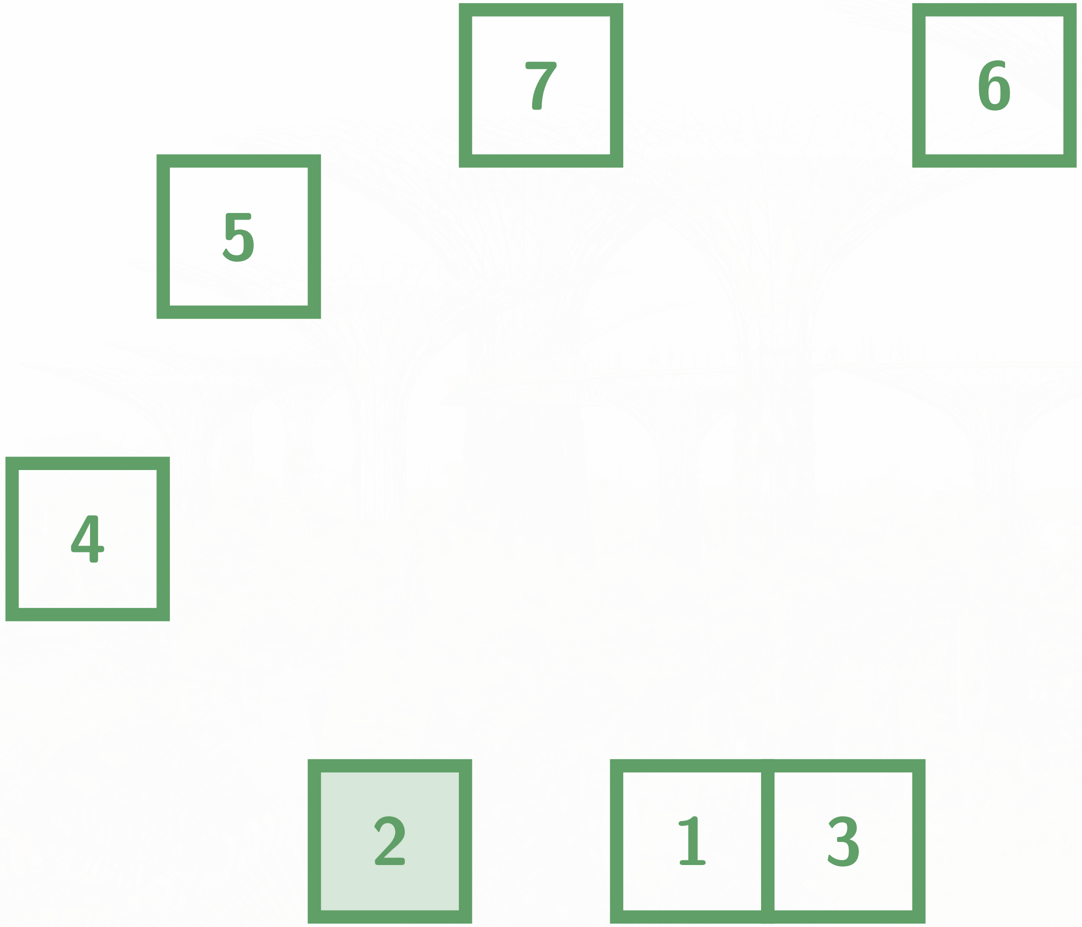

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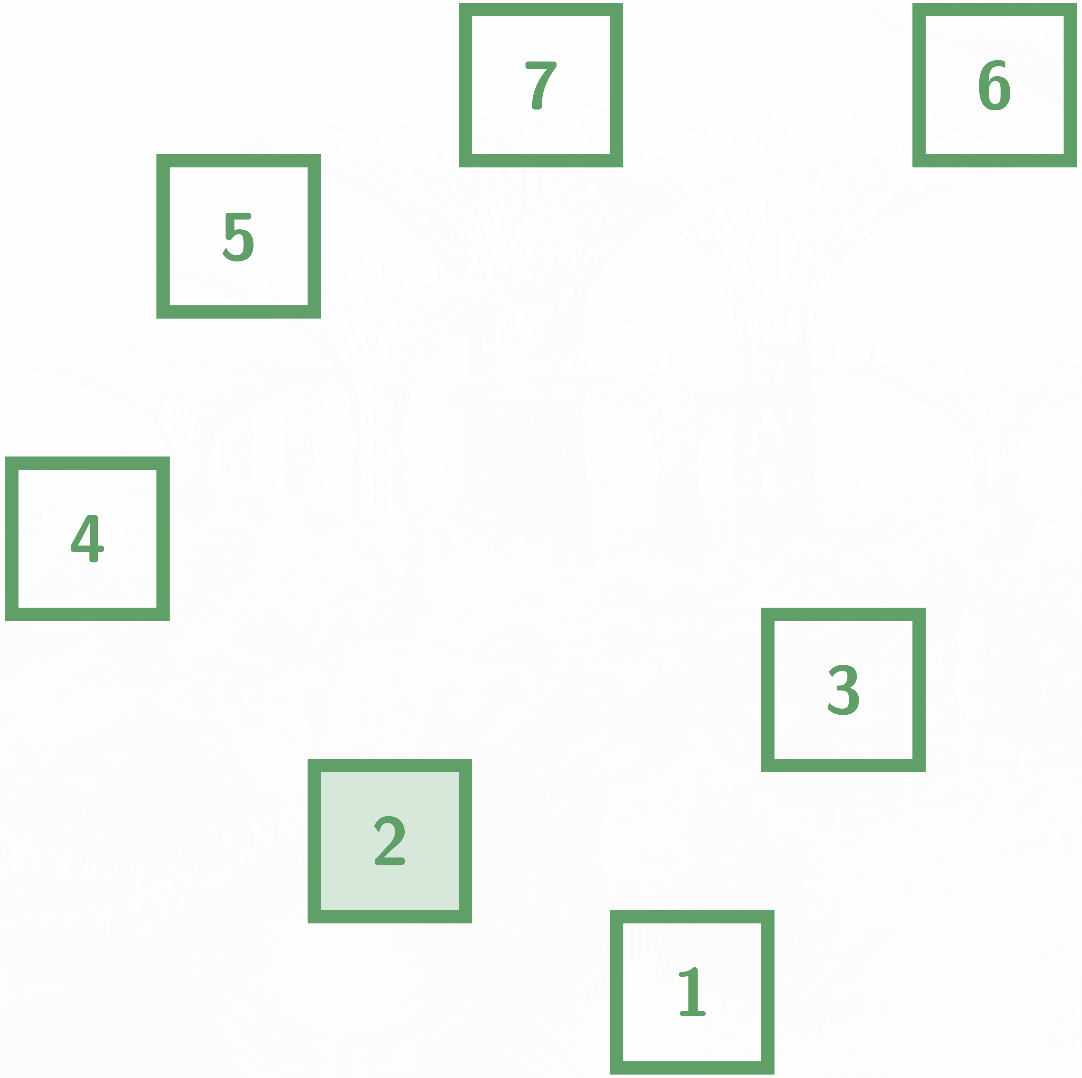

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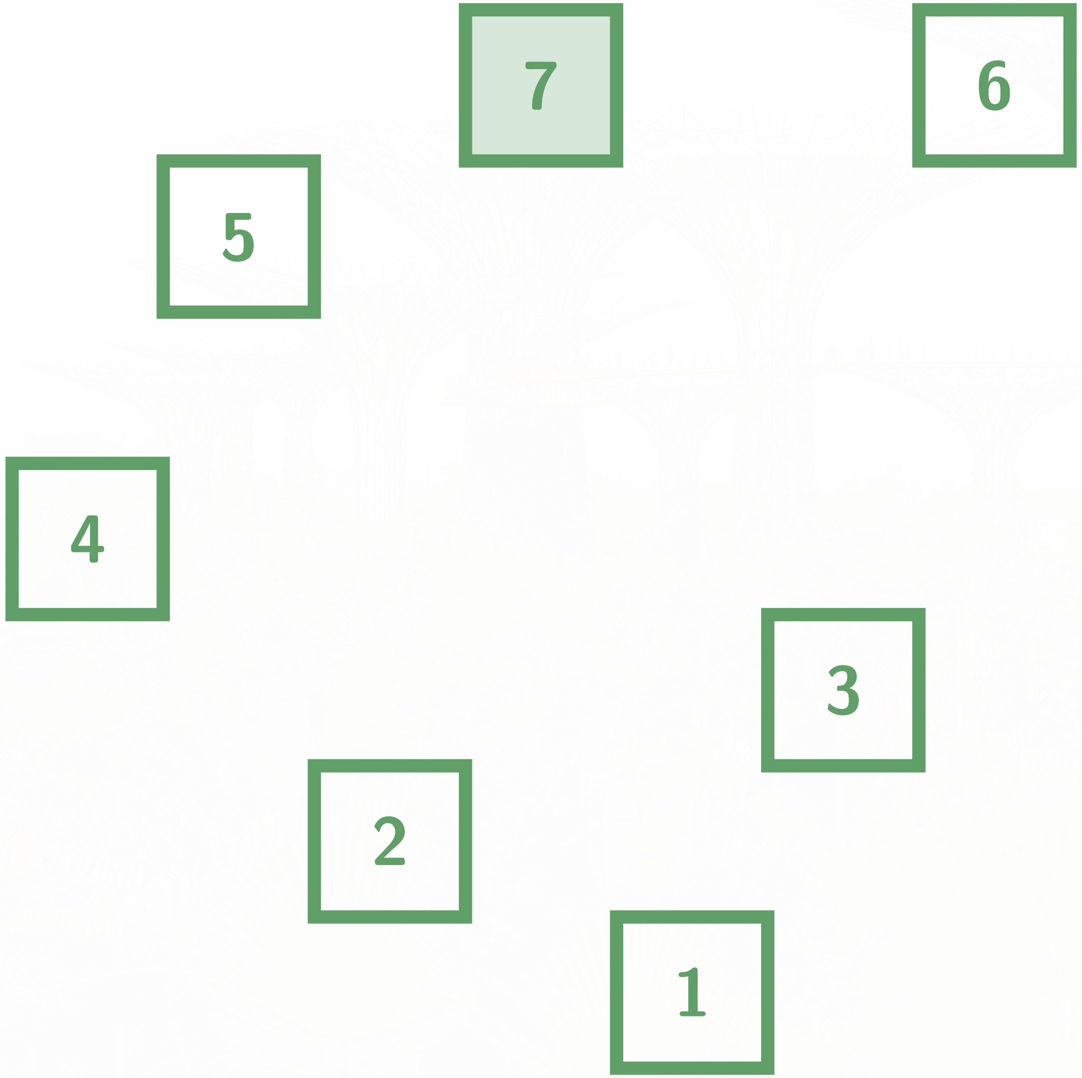

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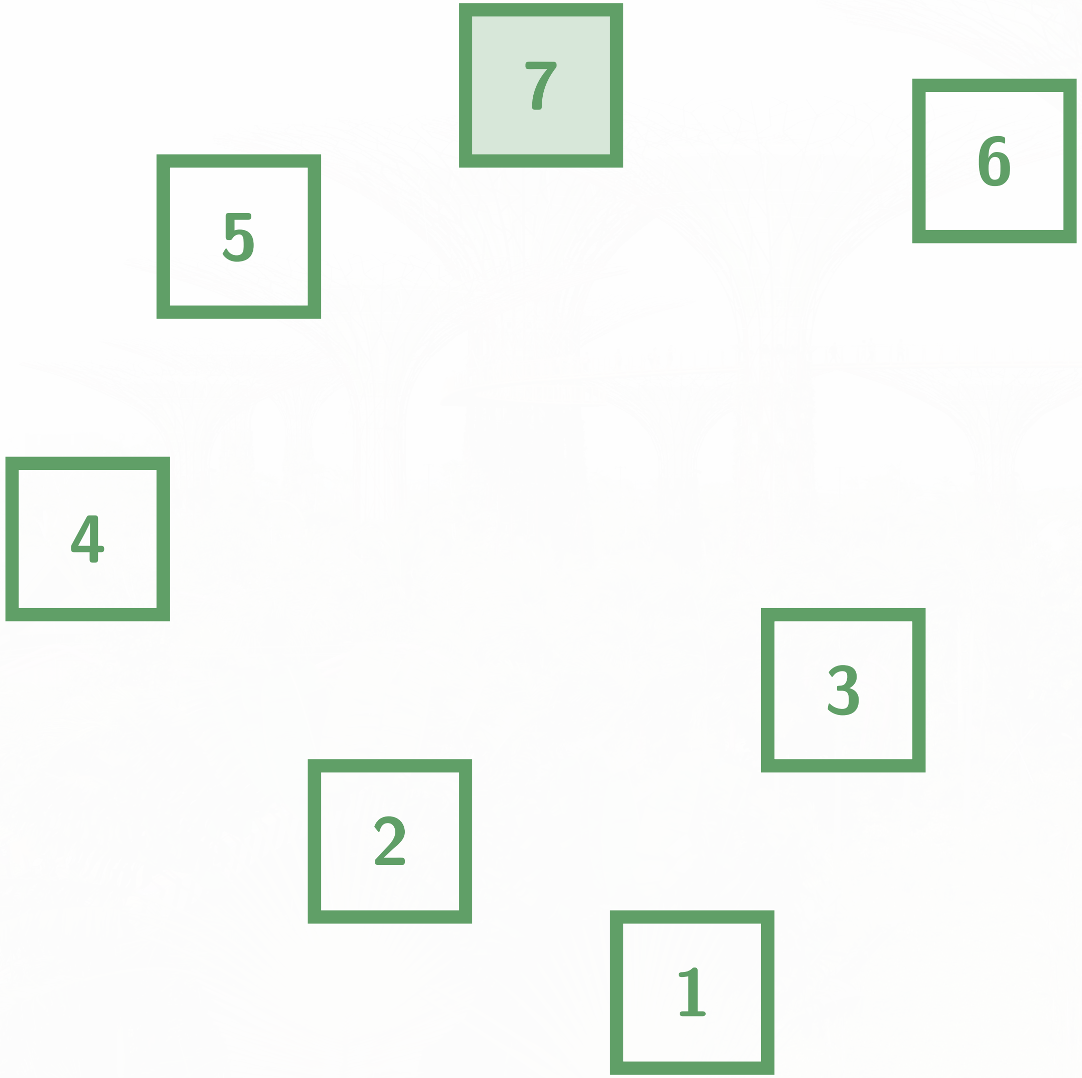

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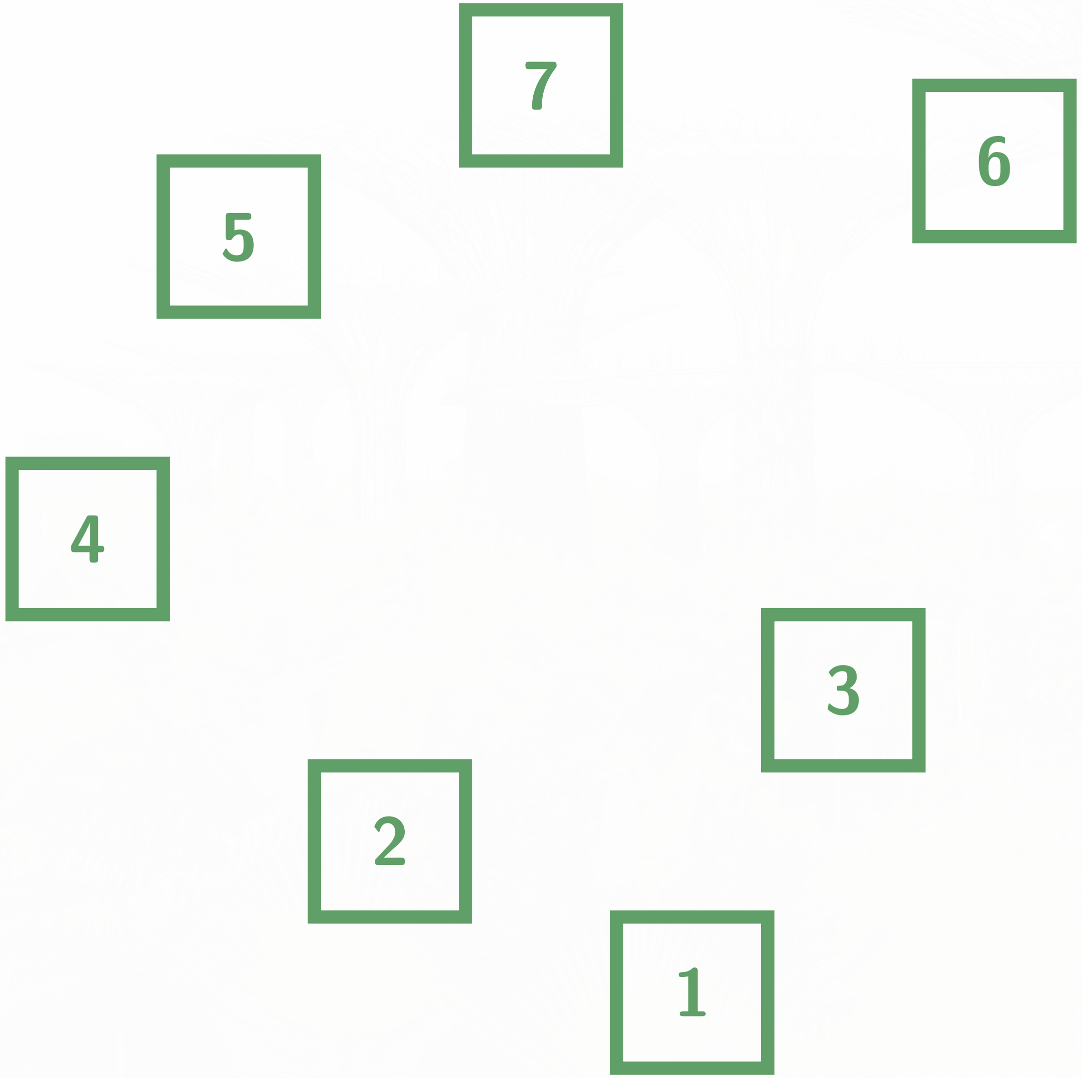

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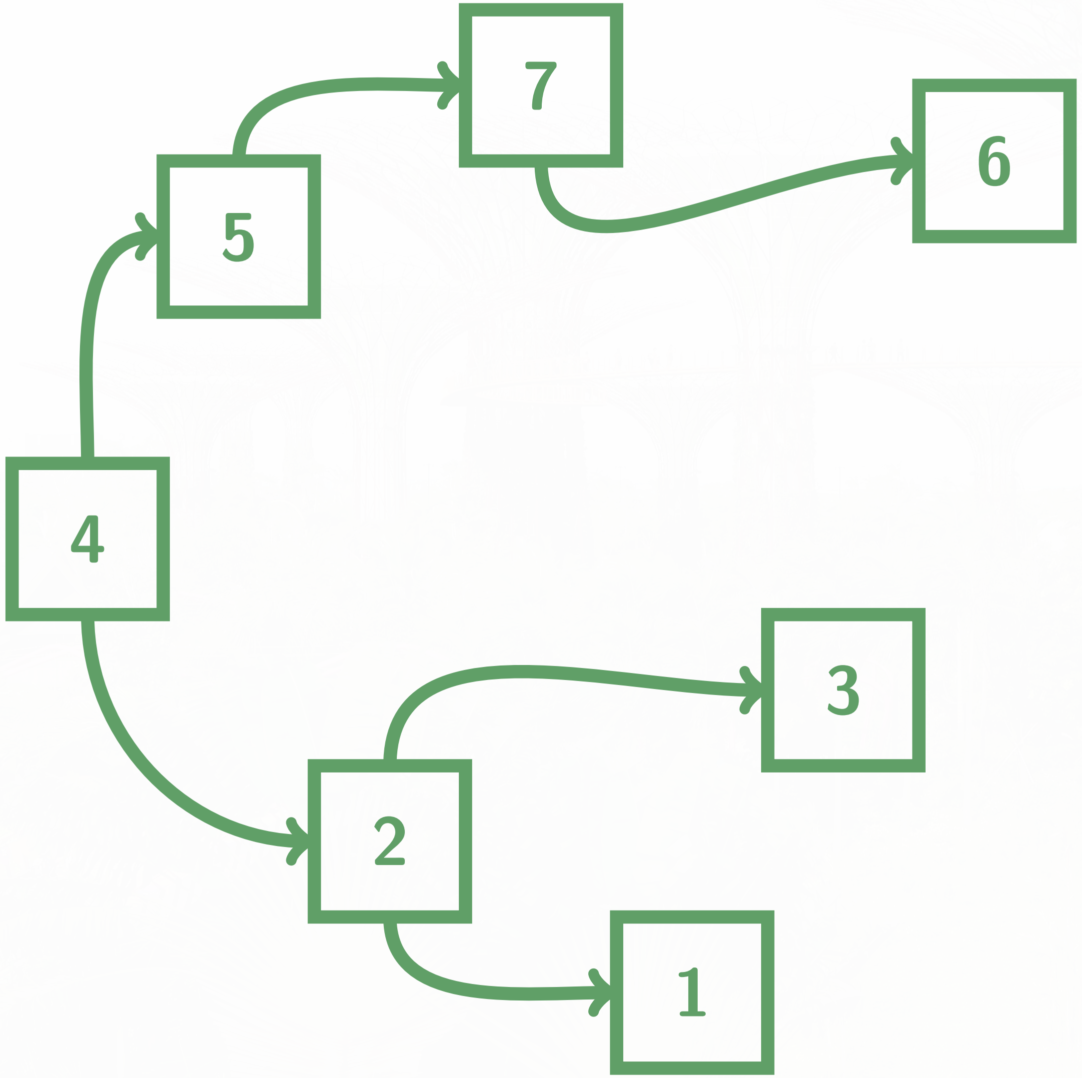

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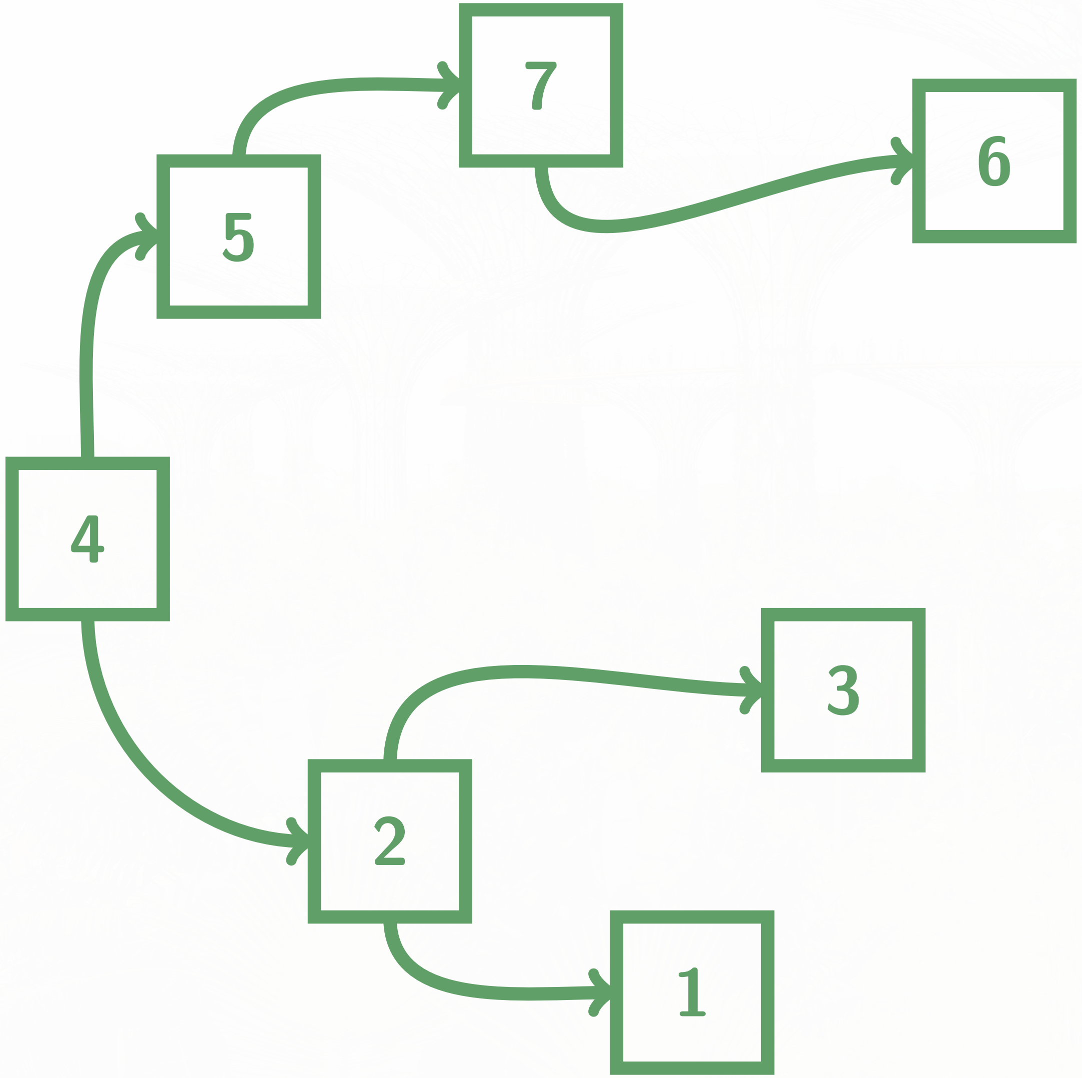

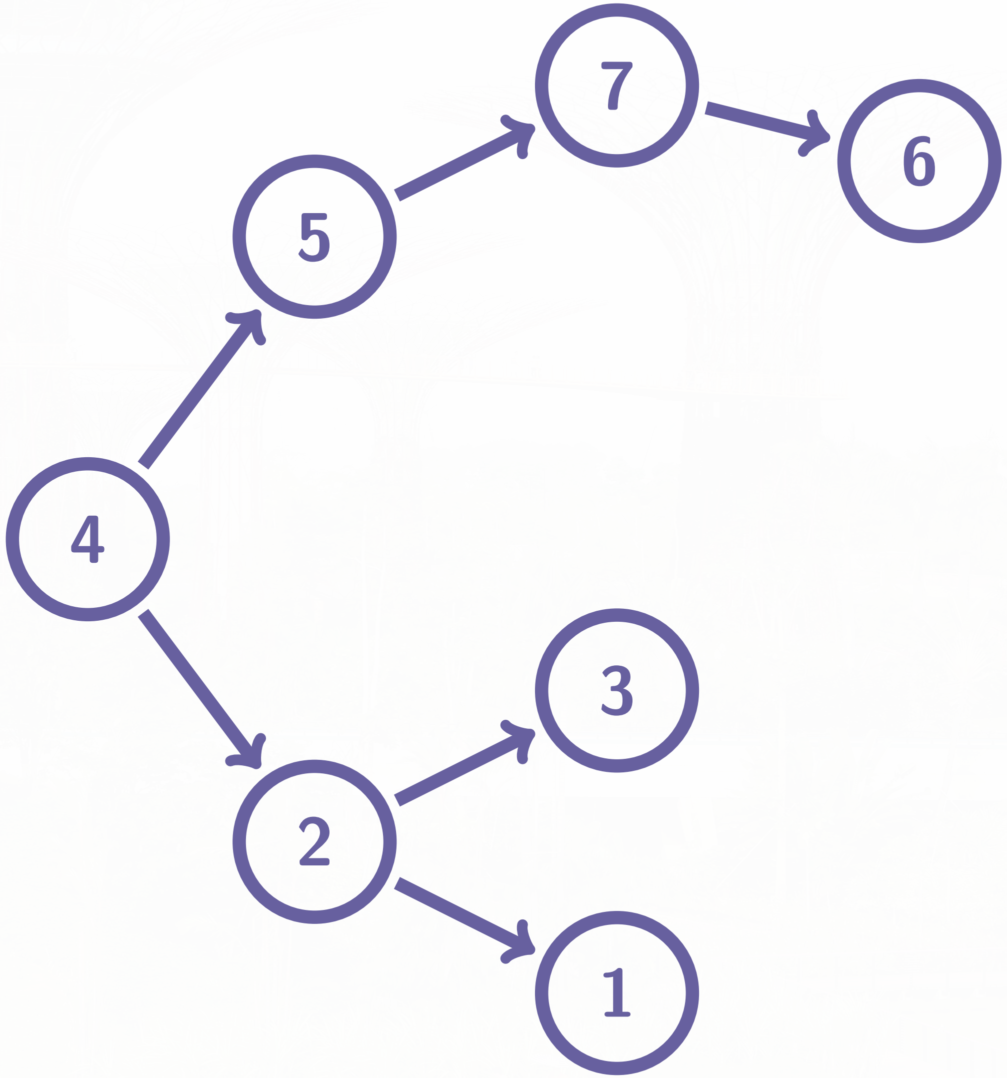

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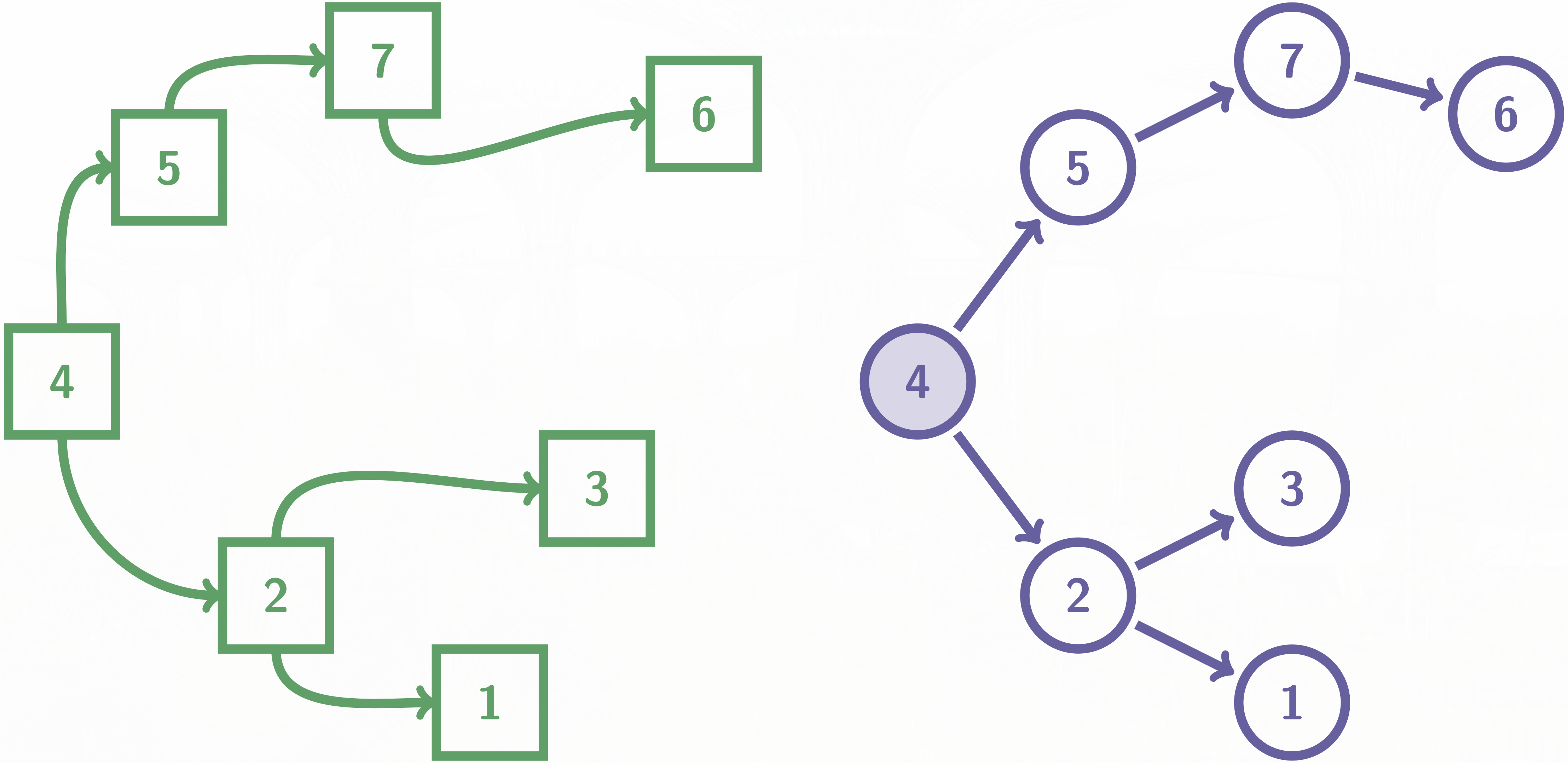

Binary search trees

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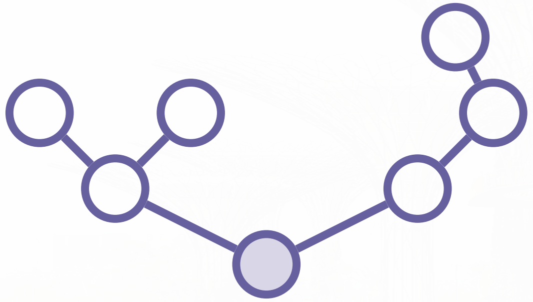

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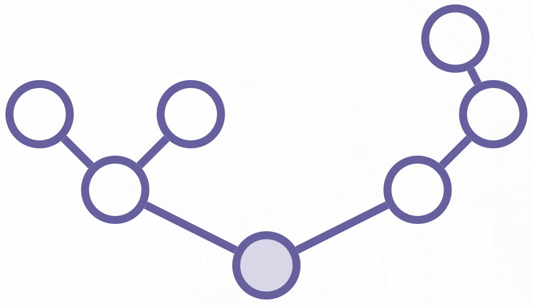

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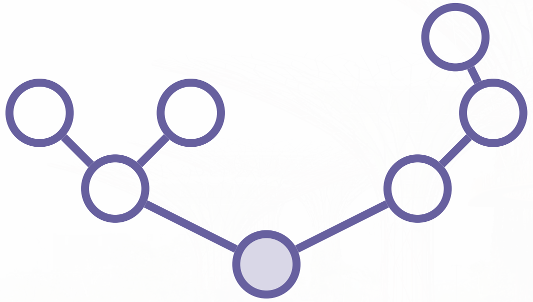

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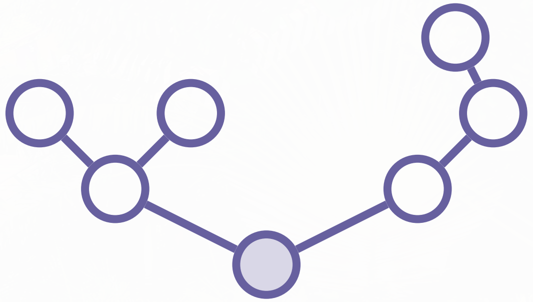

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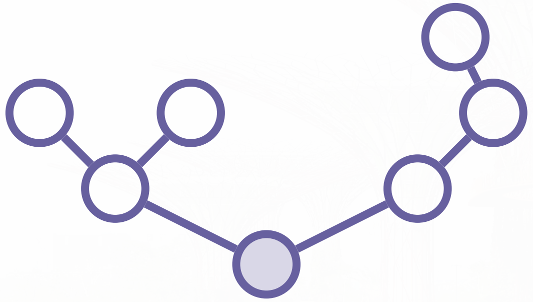

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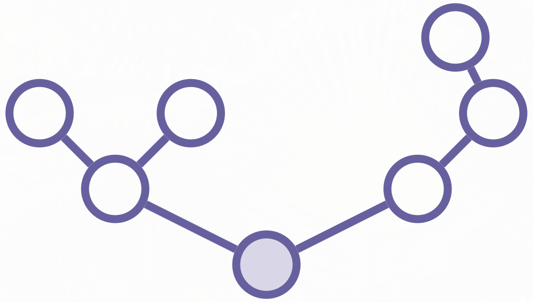

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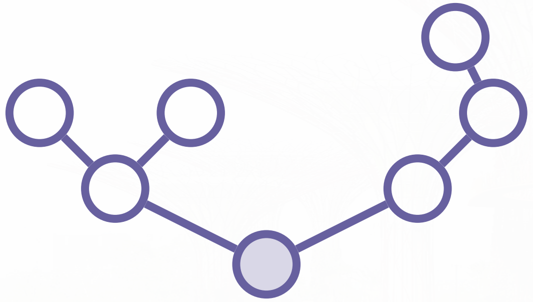

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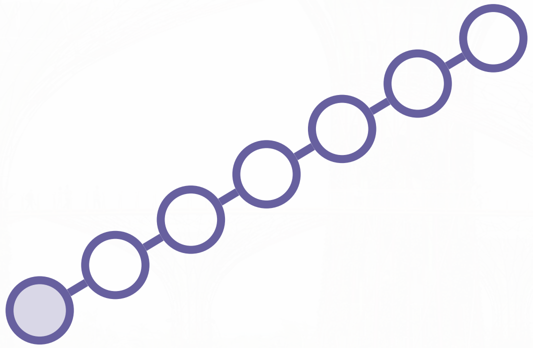

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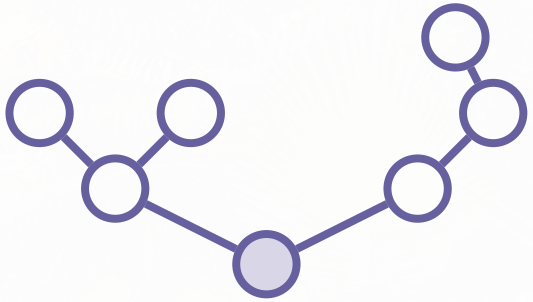

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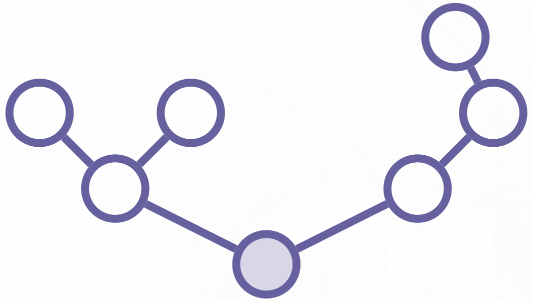

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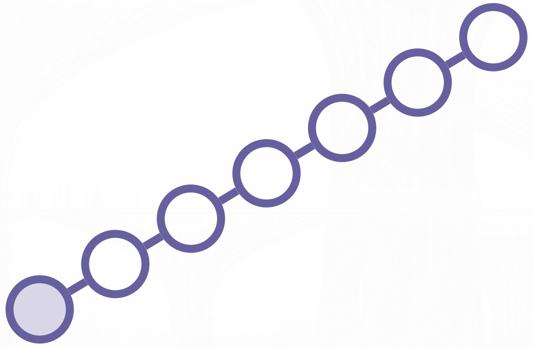

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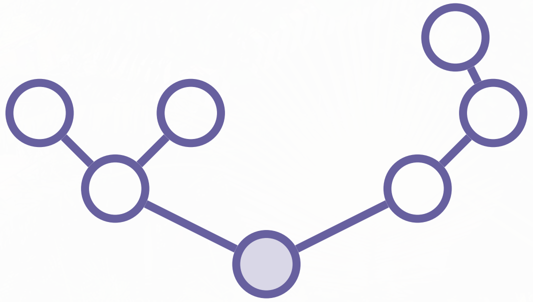

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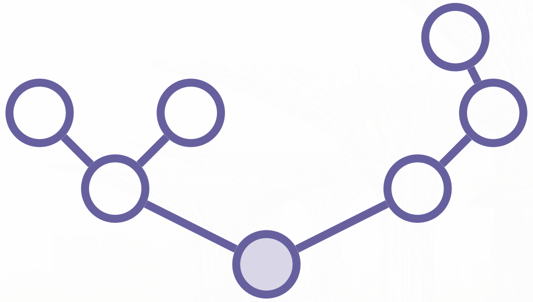

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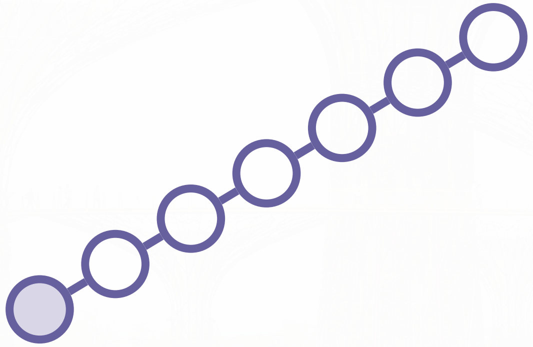

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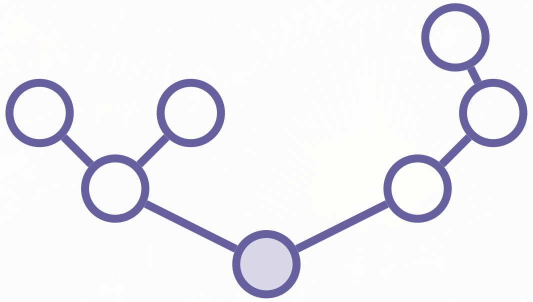

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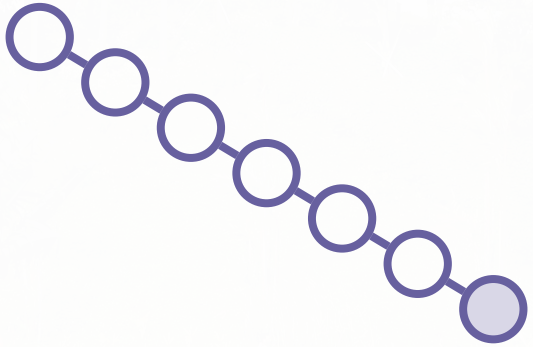

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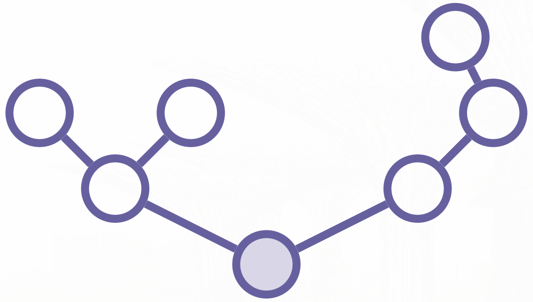

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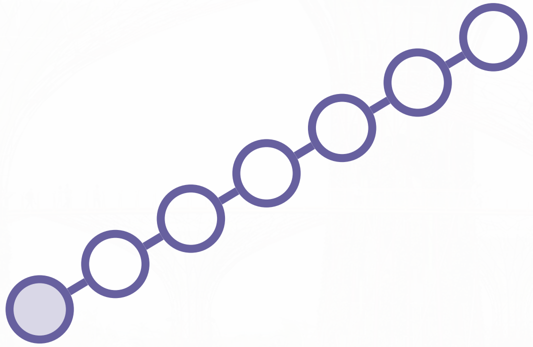

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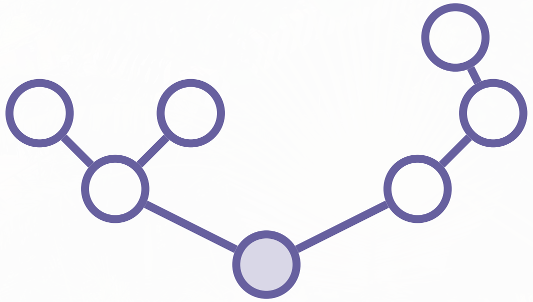

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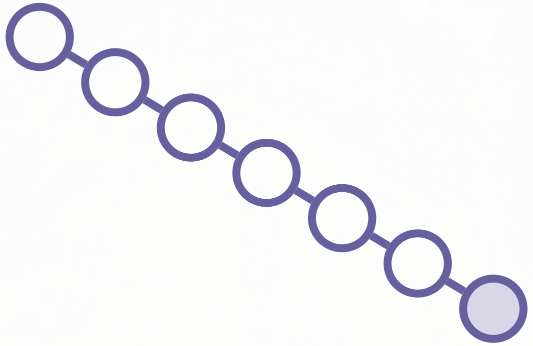

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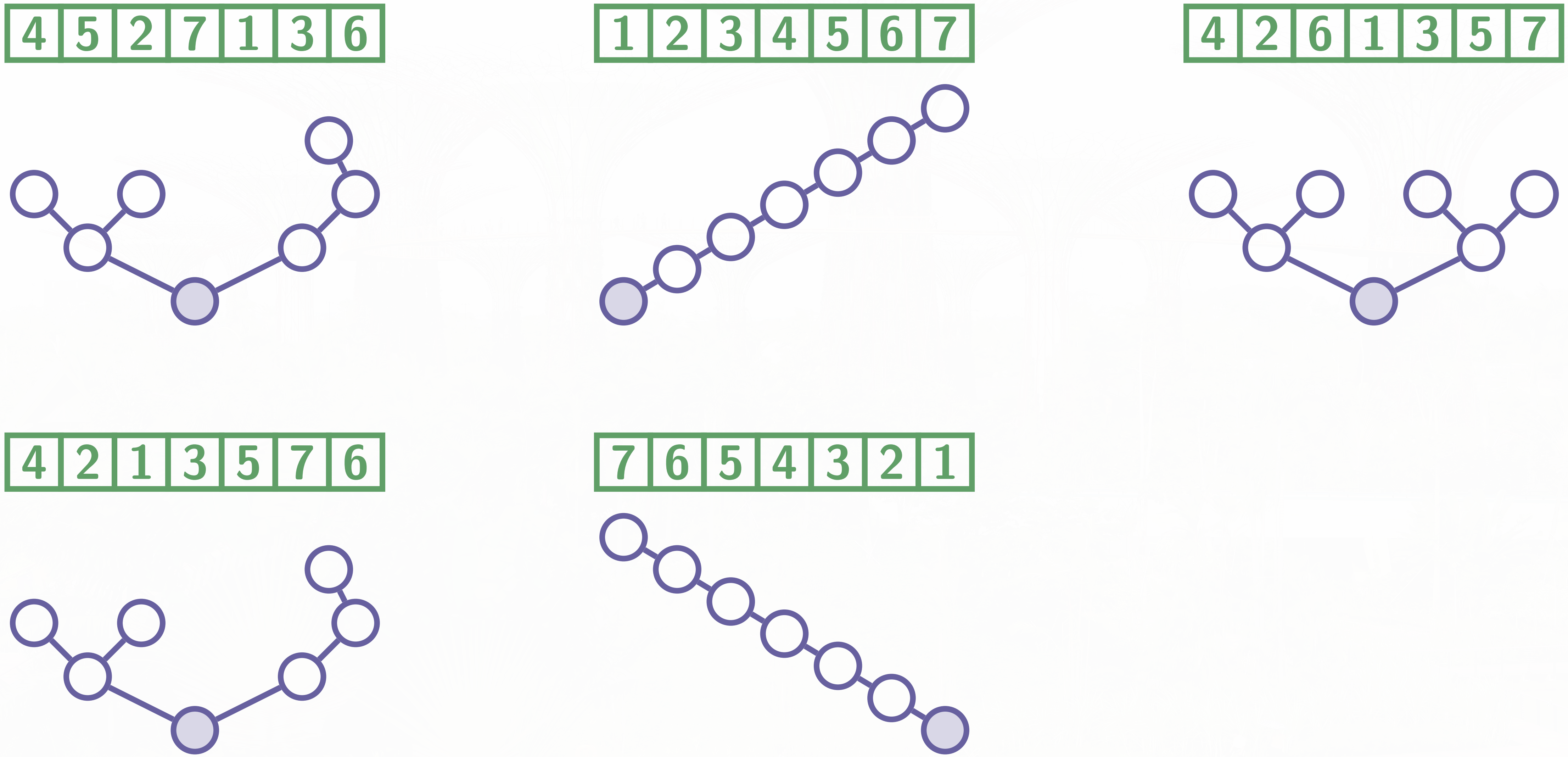

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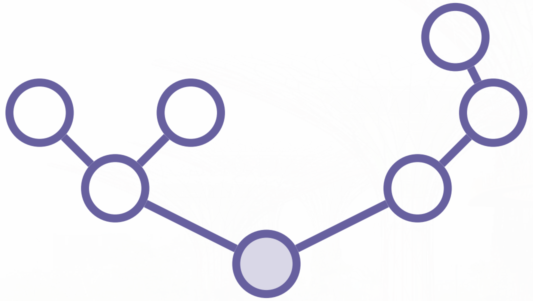

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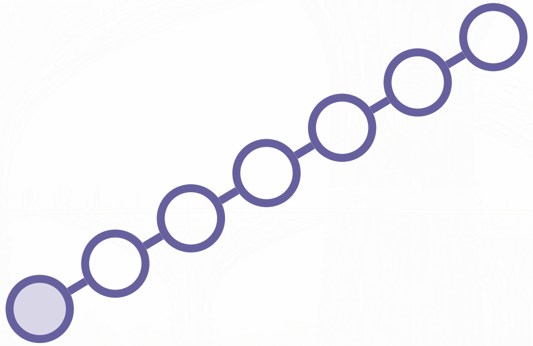

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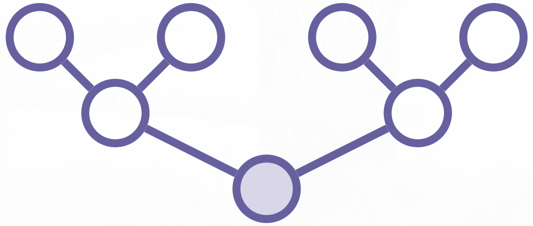

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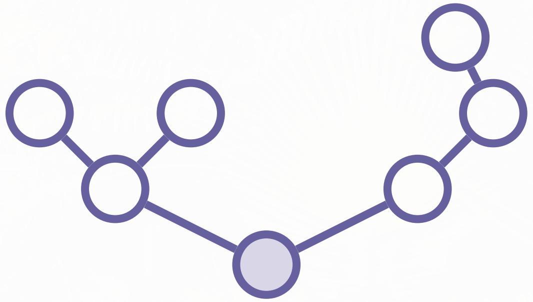

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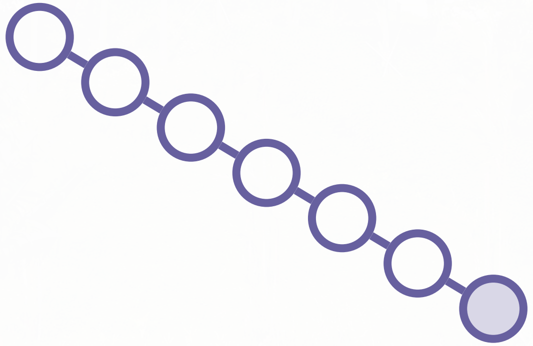

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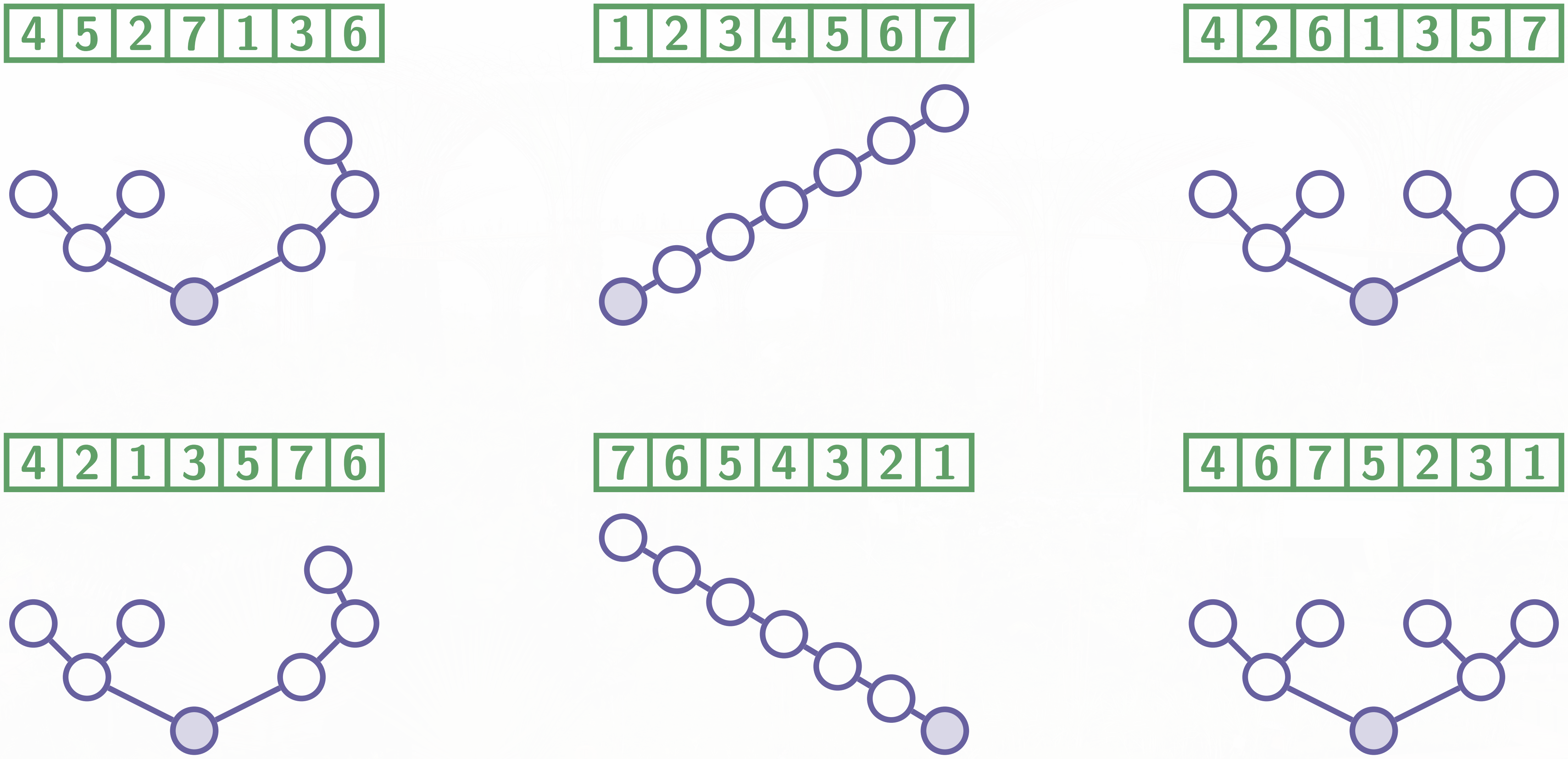

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- The height $H(T_\sigma)$ is related to the efficiency of the quicksort algorithm applied to σ.
- For any σ of length n, we have that $\lfloor \log_2 n \rfloor \leq h(T_\sigma) \leq n - 1$.

Random binary search trees

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Theorem (Devroye, 1986)

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Let σ be a sequence of n random (distinct) values. Then, as n goes to infinity,

$$\frac{h(T_\sigma)}{\log n} \longrightarrow c^*,$$

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Note: It is of the same order of magnitude as the minimal height: $\lfloor \log_2 n \rfloor \simeq 1.443 \cdot \log n$.

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Q: Is $c^* \log n$ a natural/universal lower bound for the height of random sequences?

Q: Are there other interesting random models of sequences to study?

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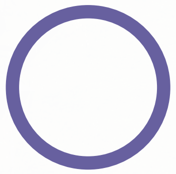

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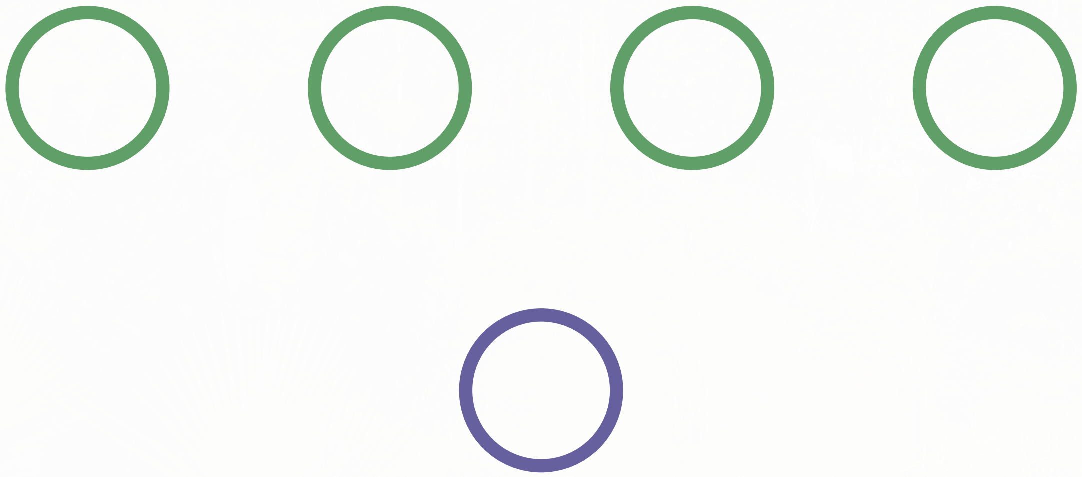

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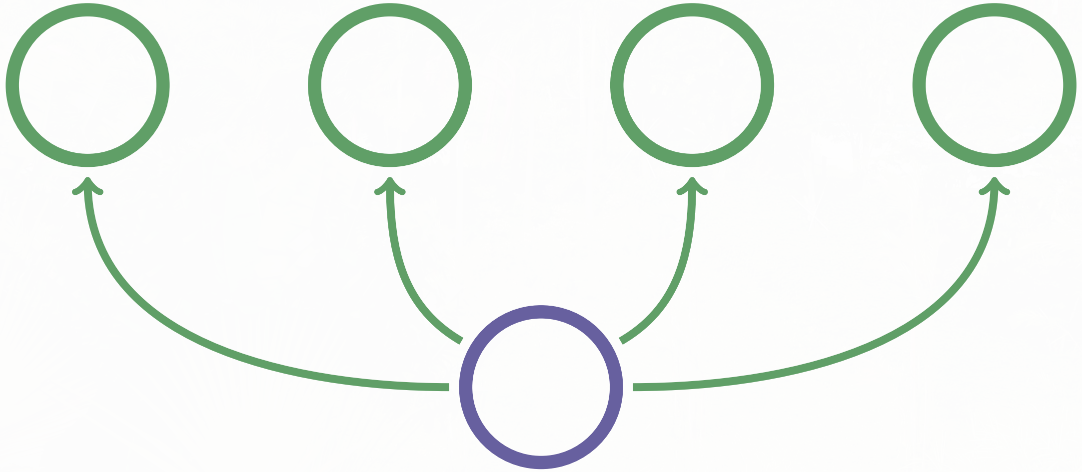

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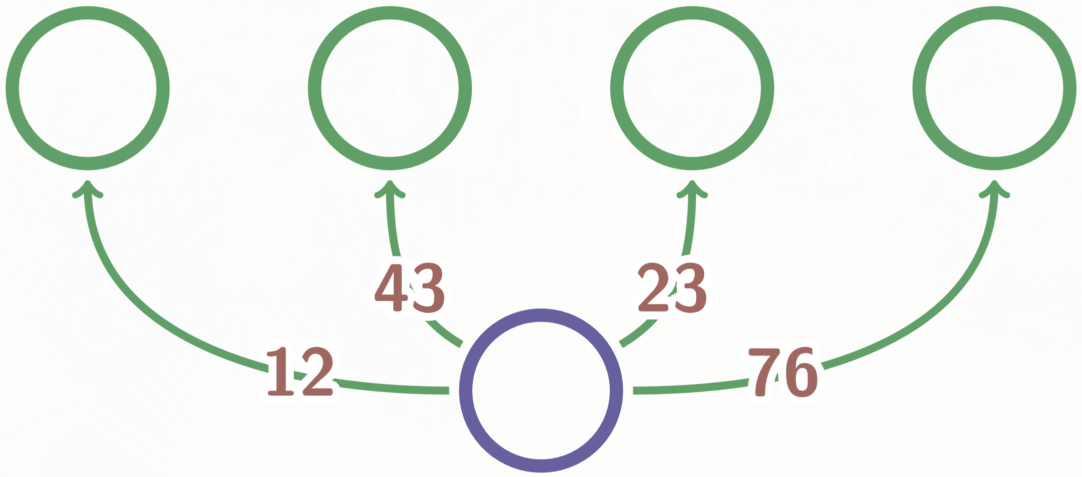

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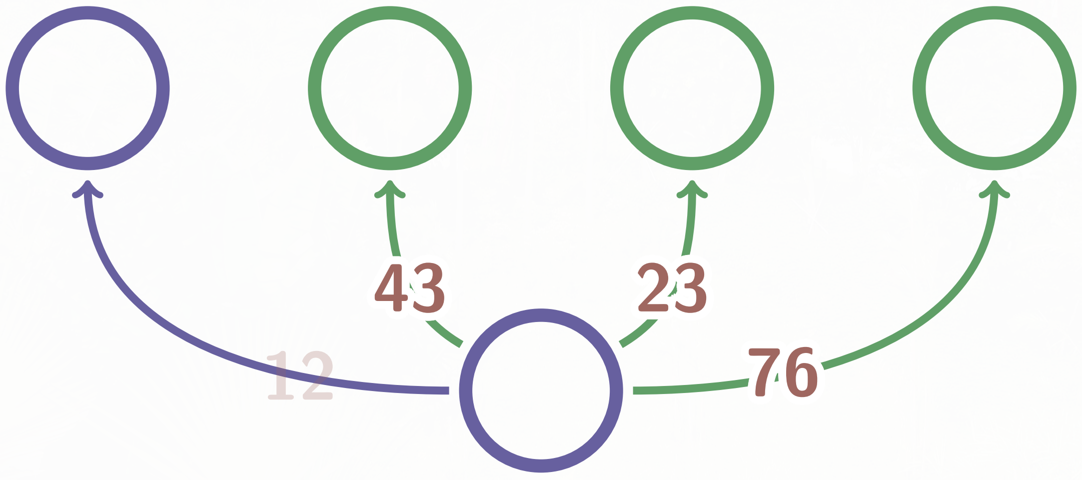

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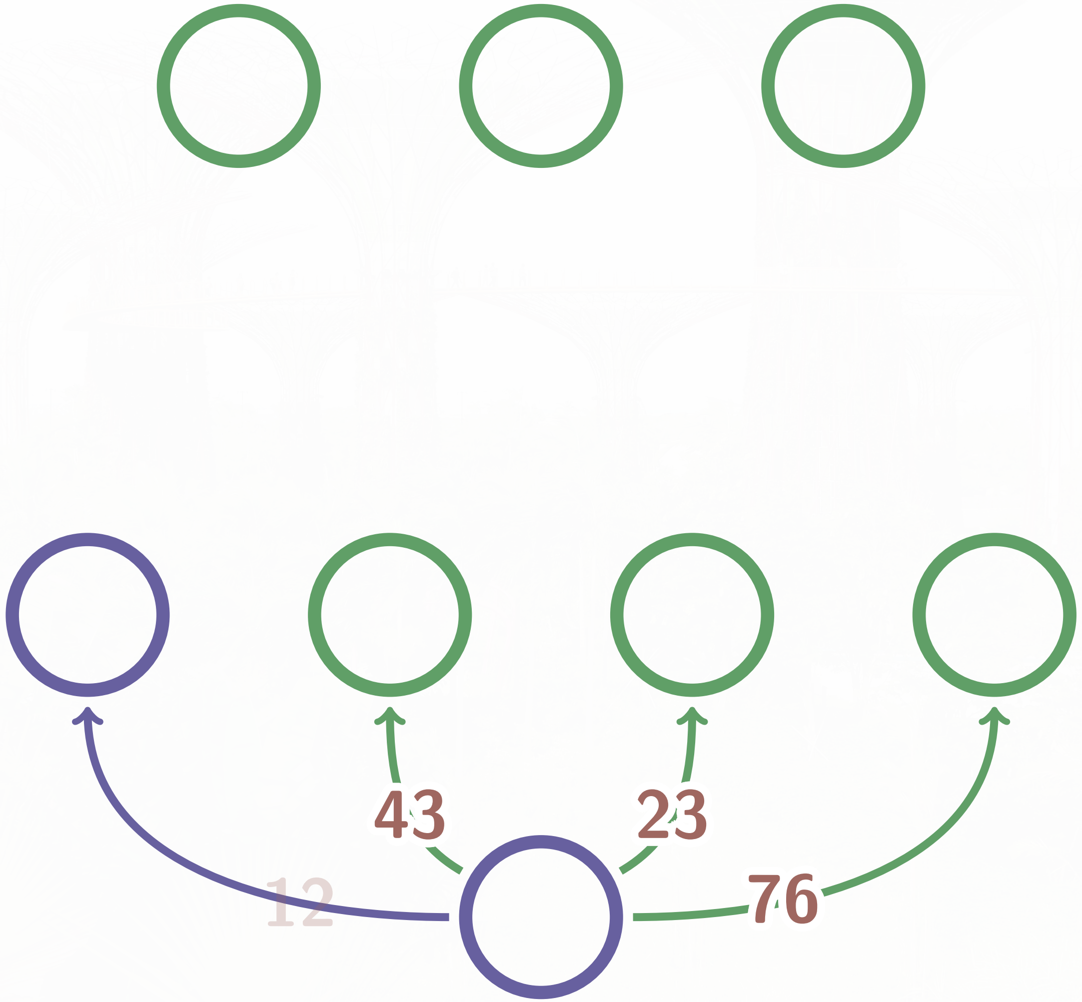

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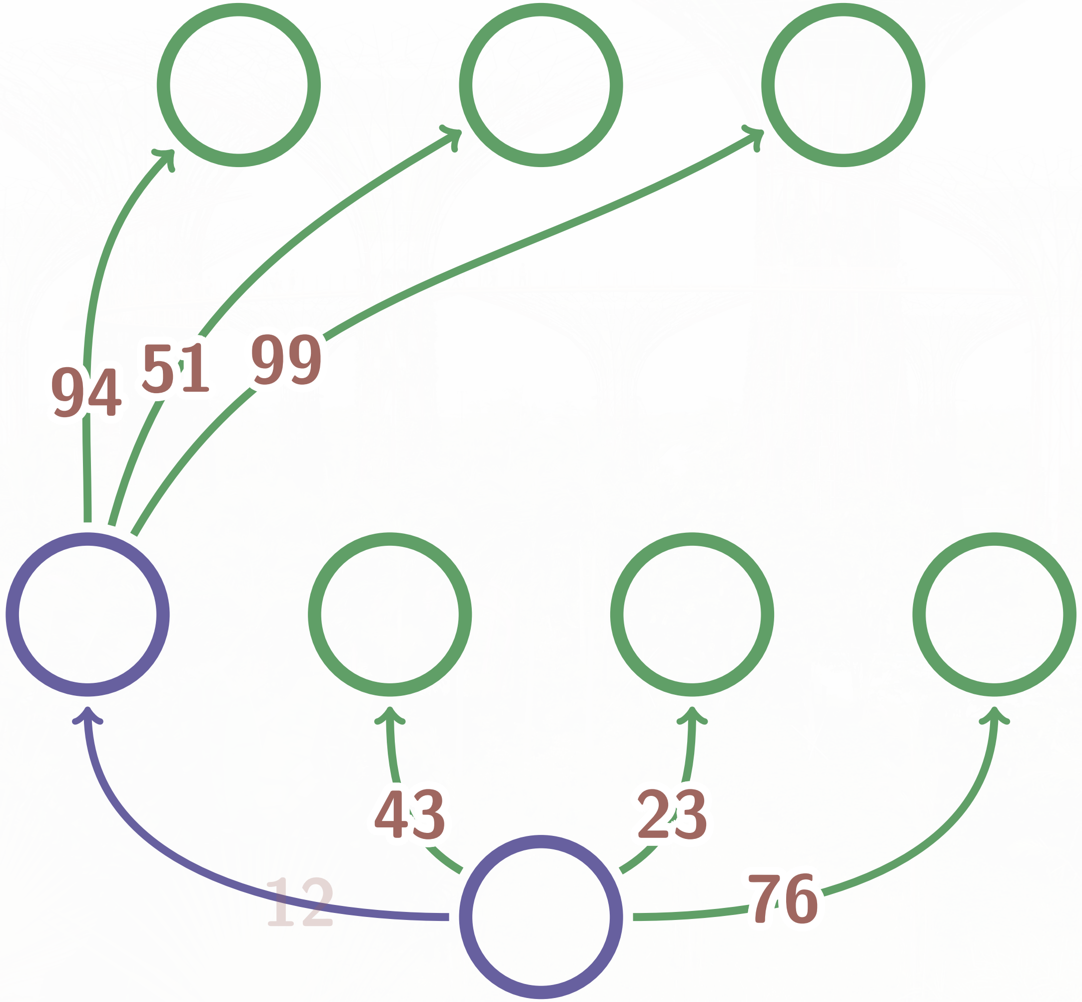

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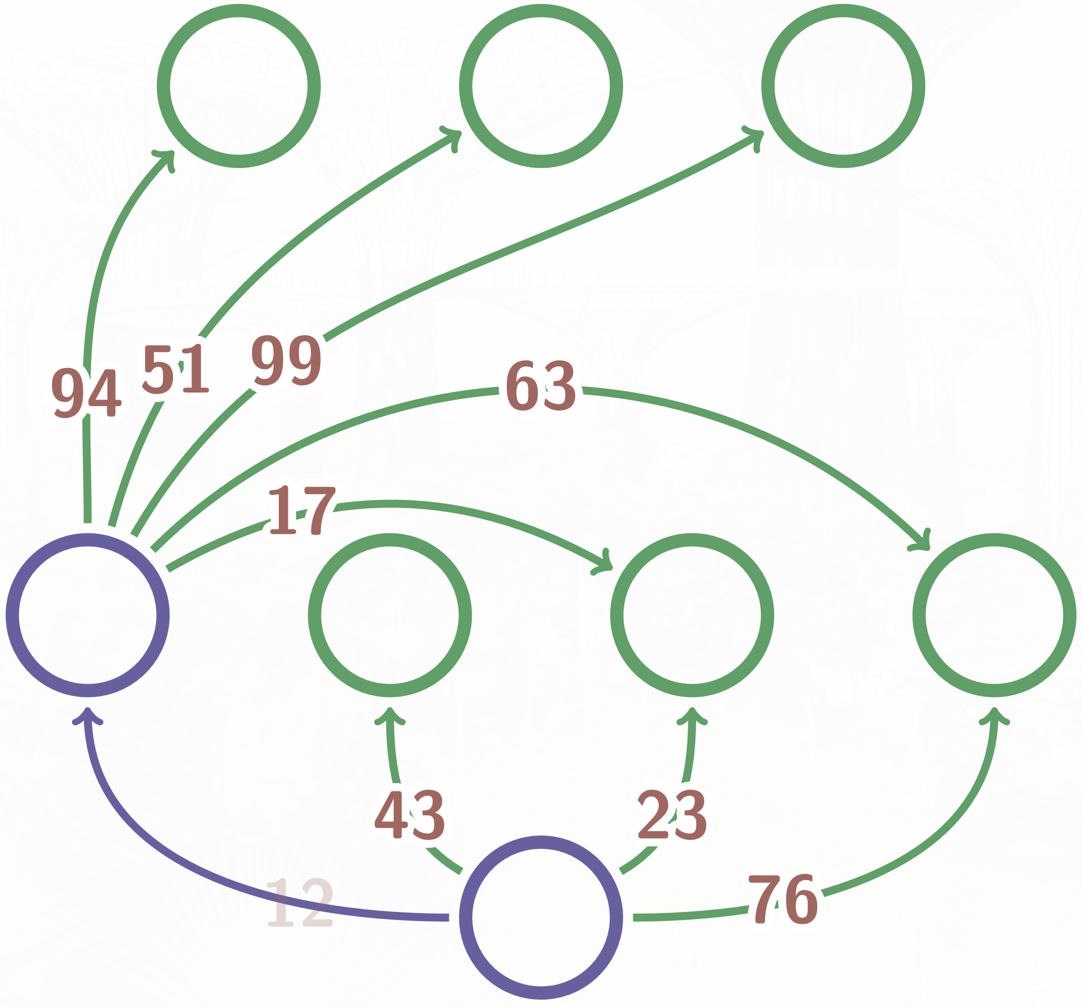

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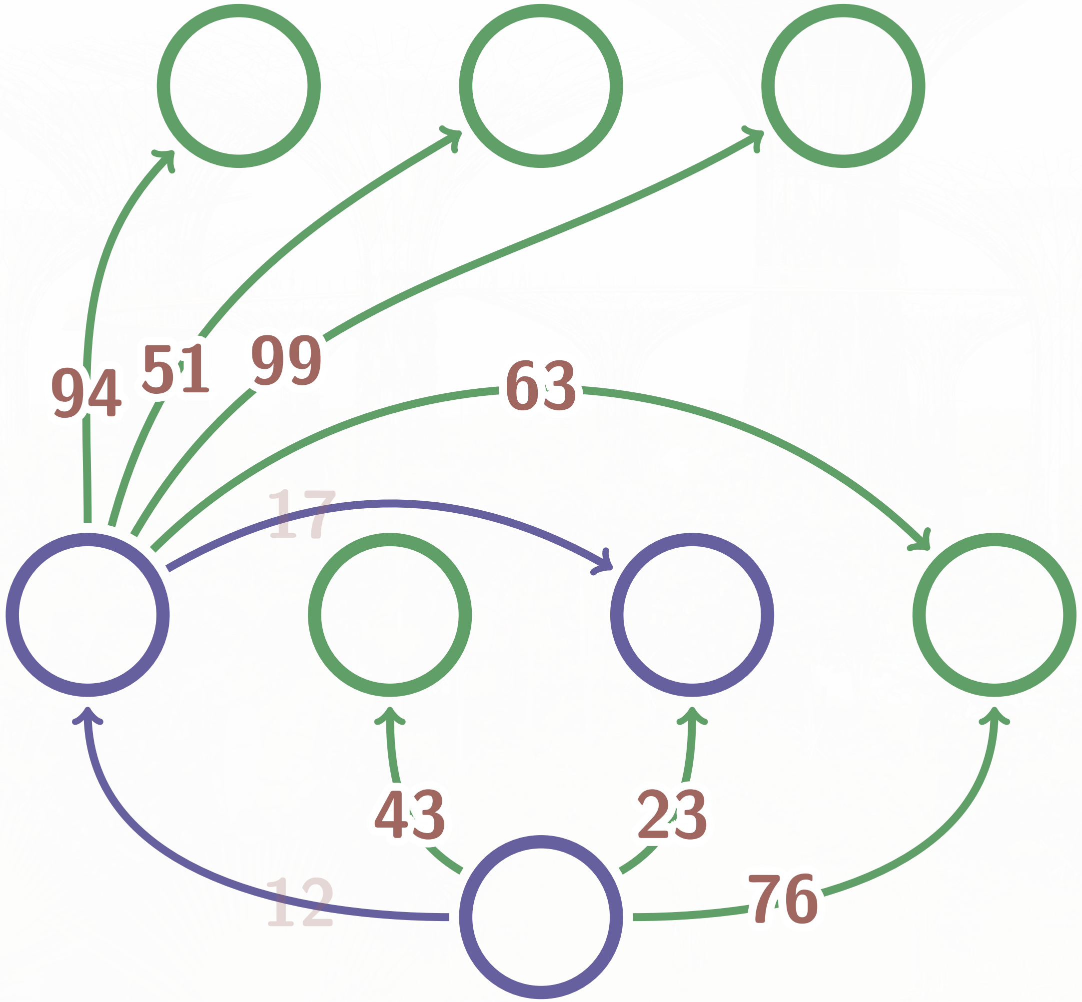

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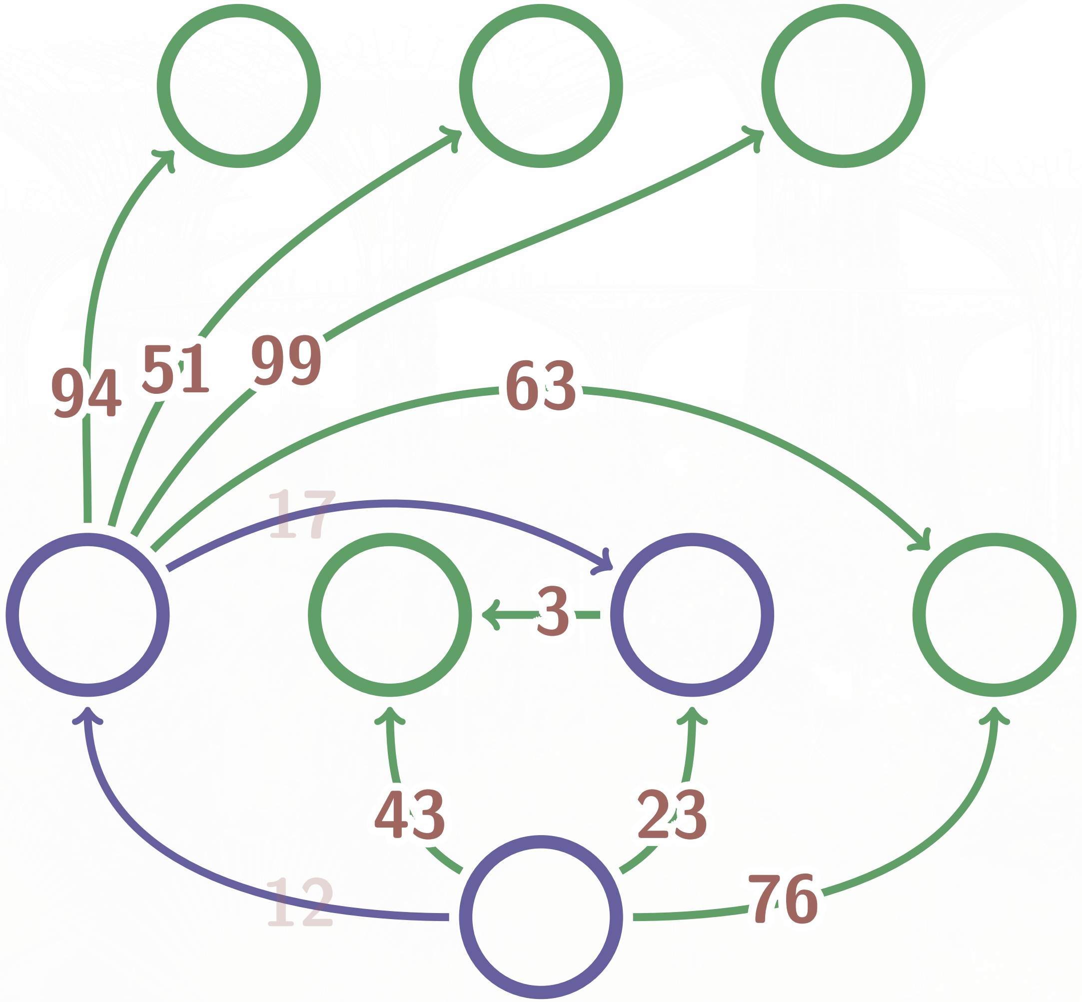

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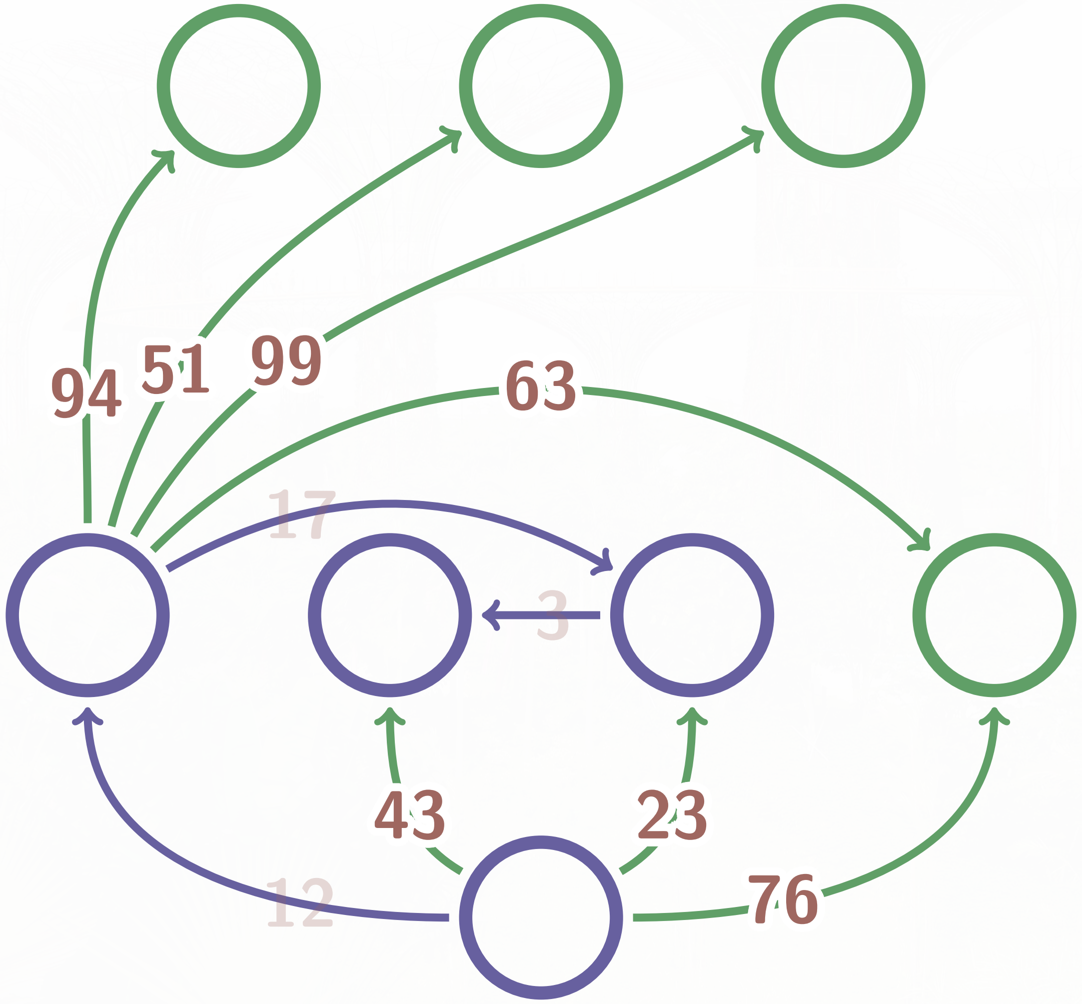

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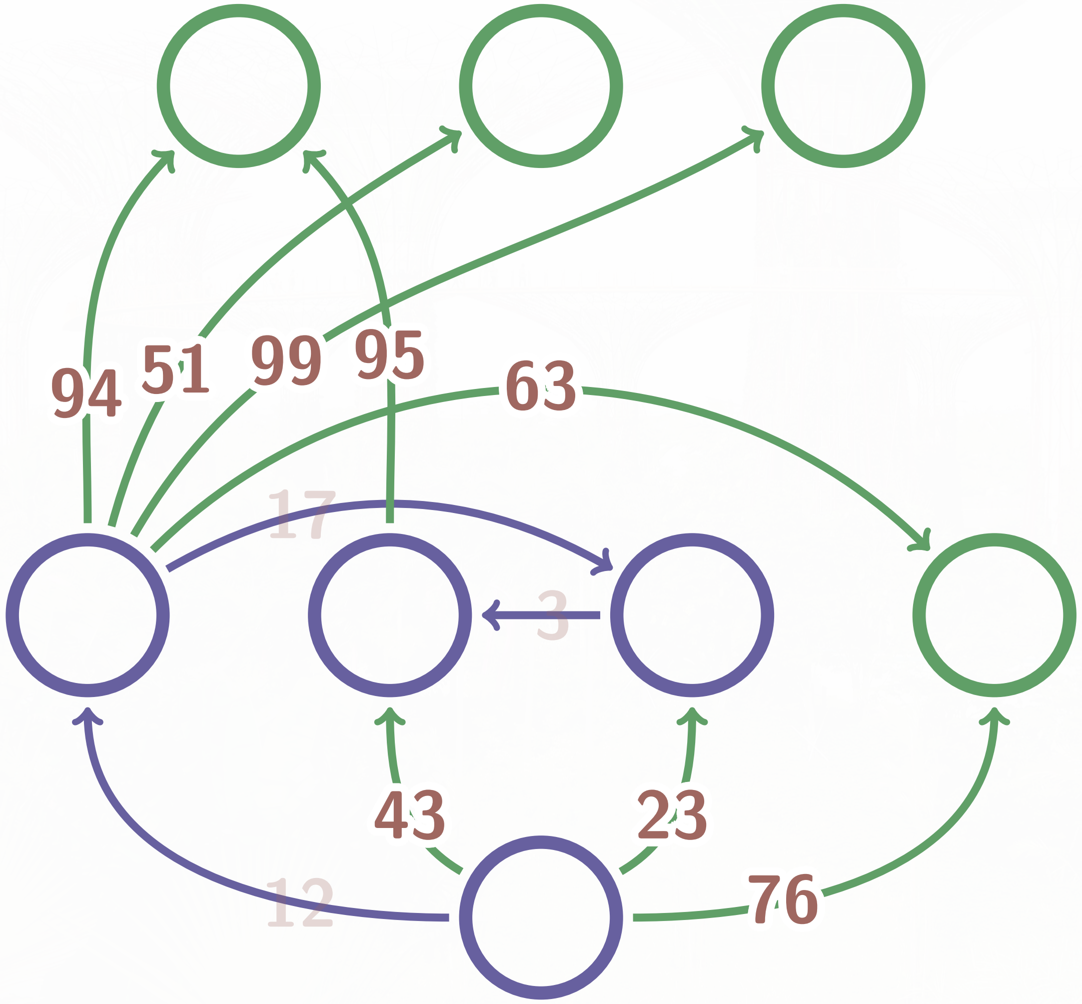

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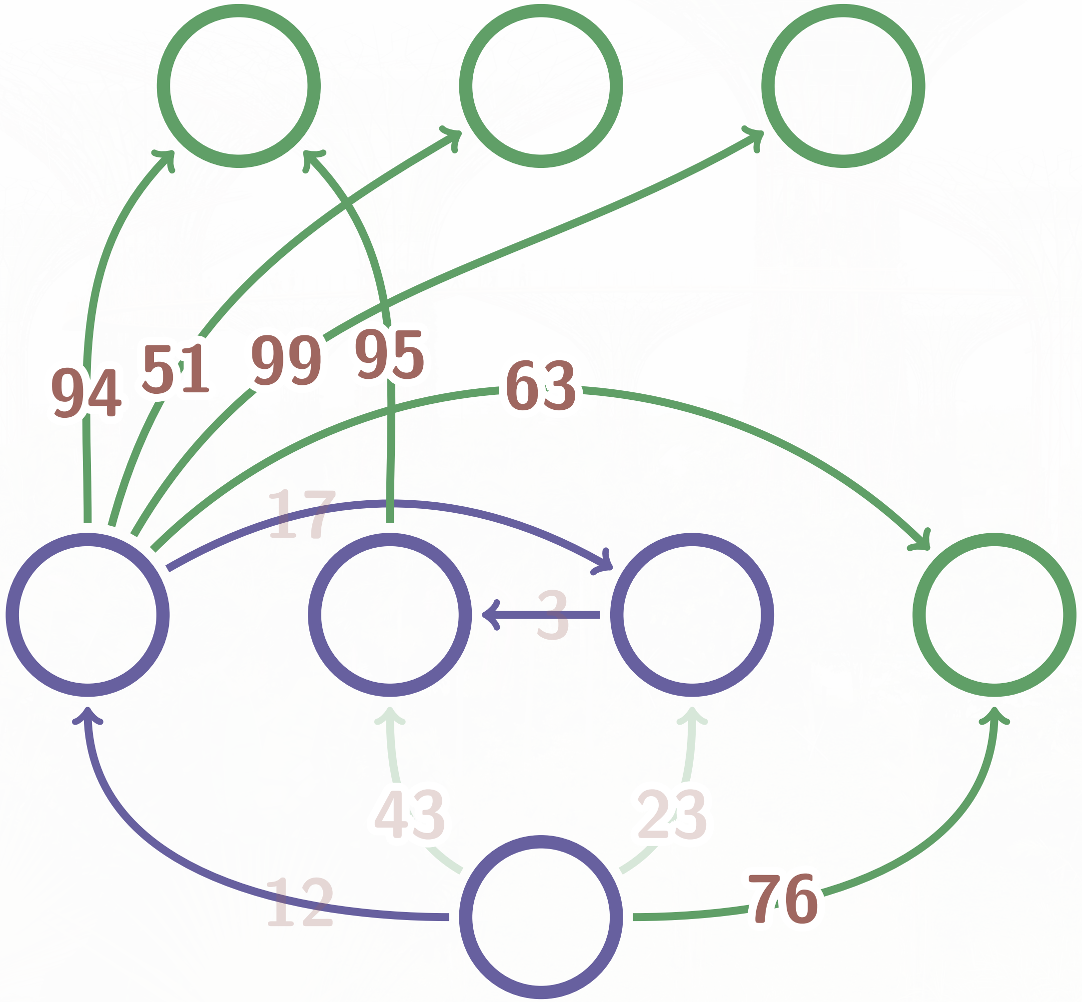

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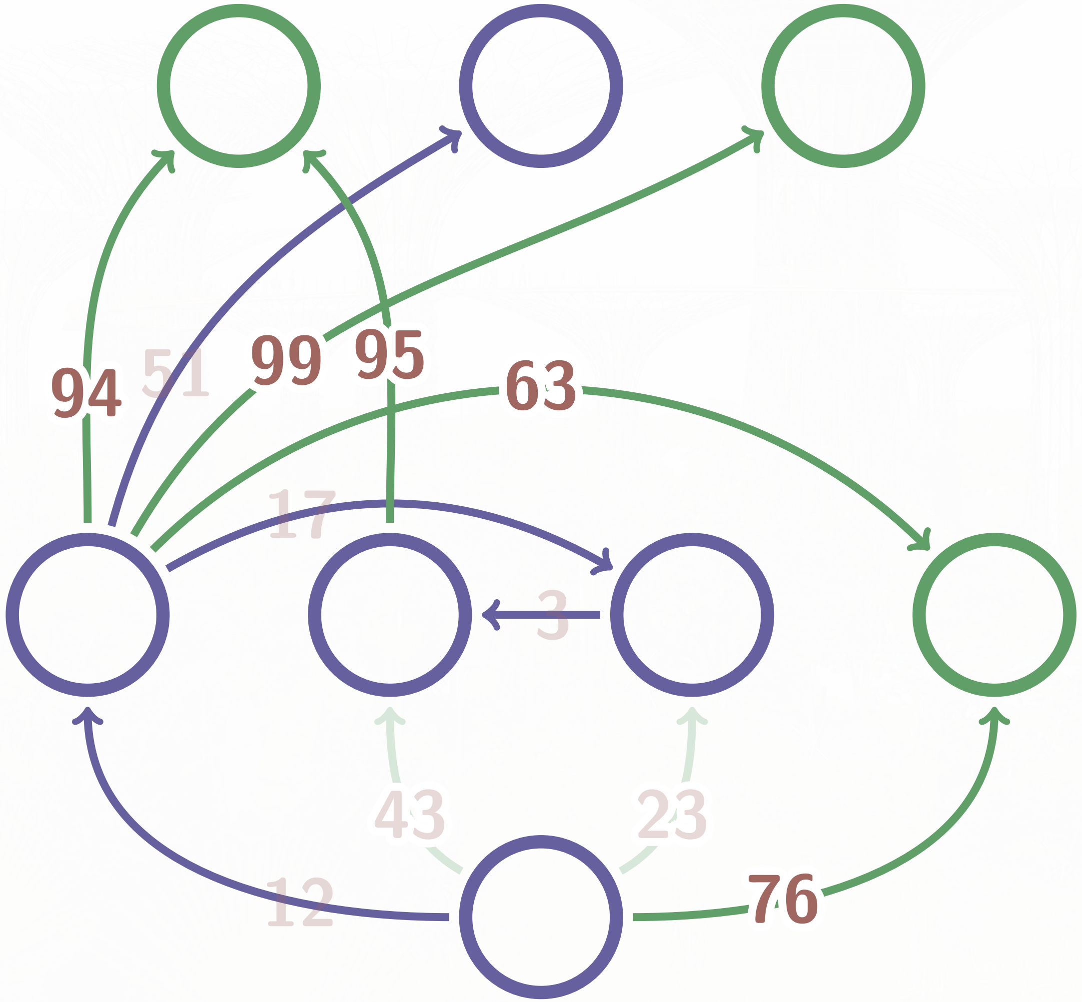

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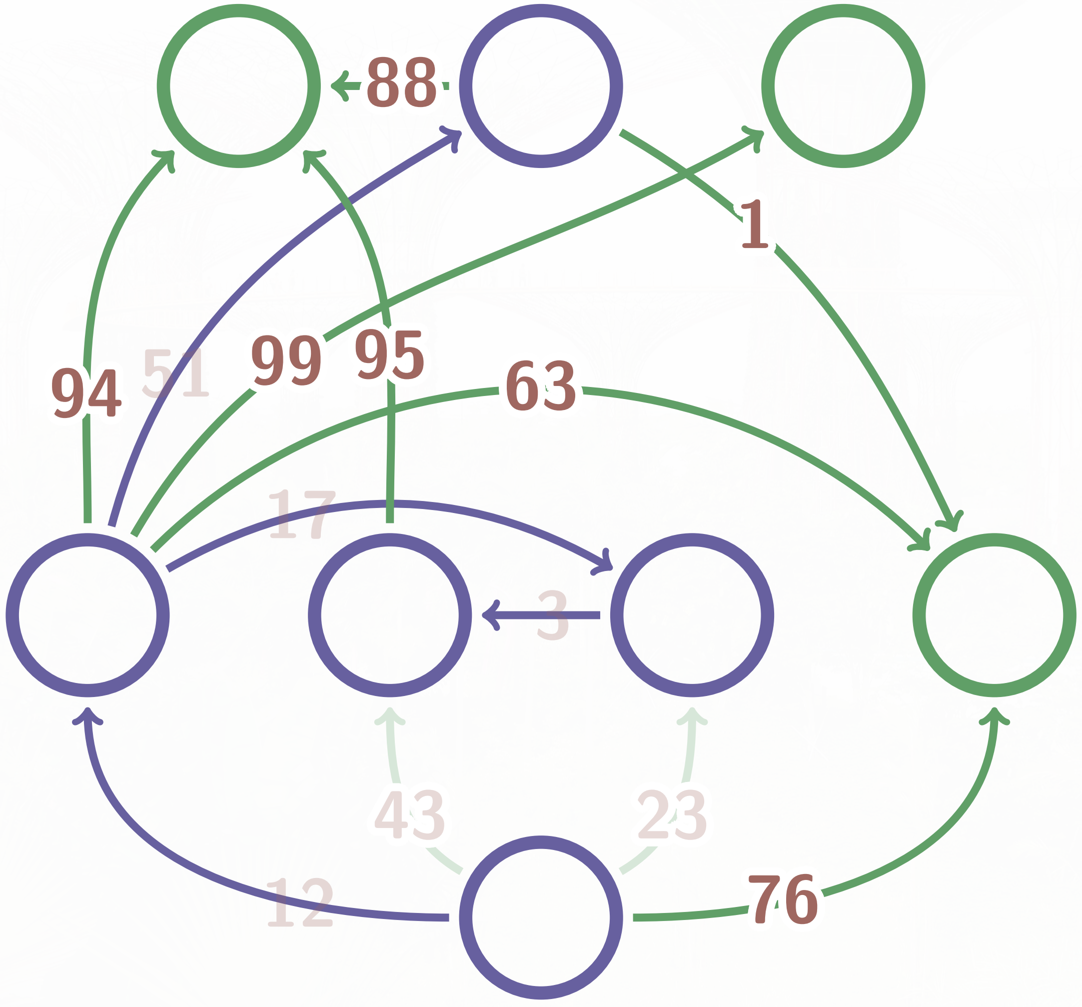

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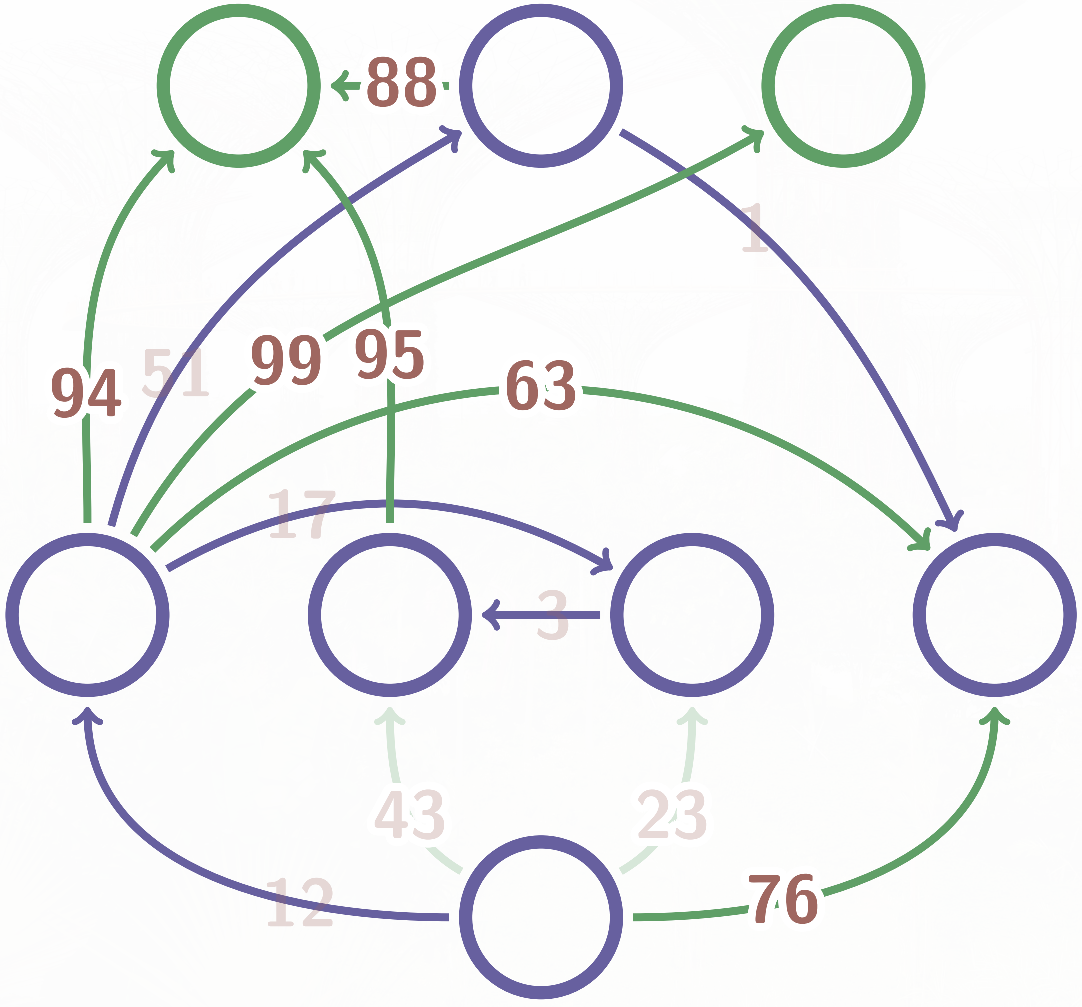

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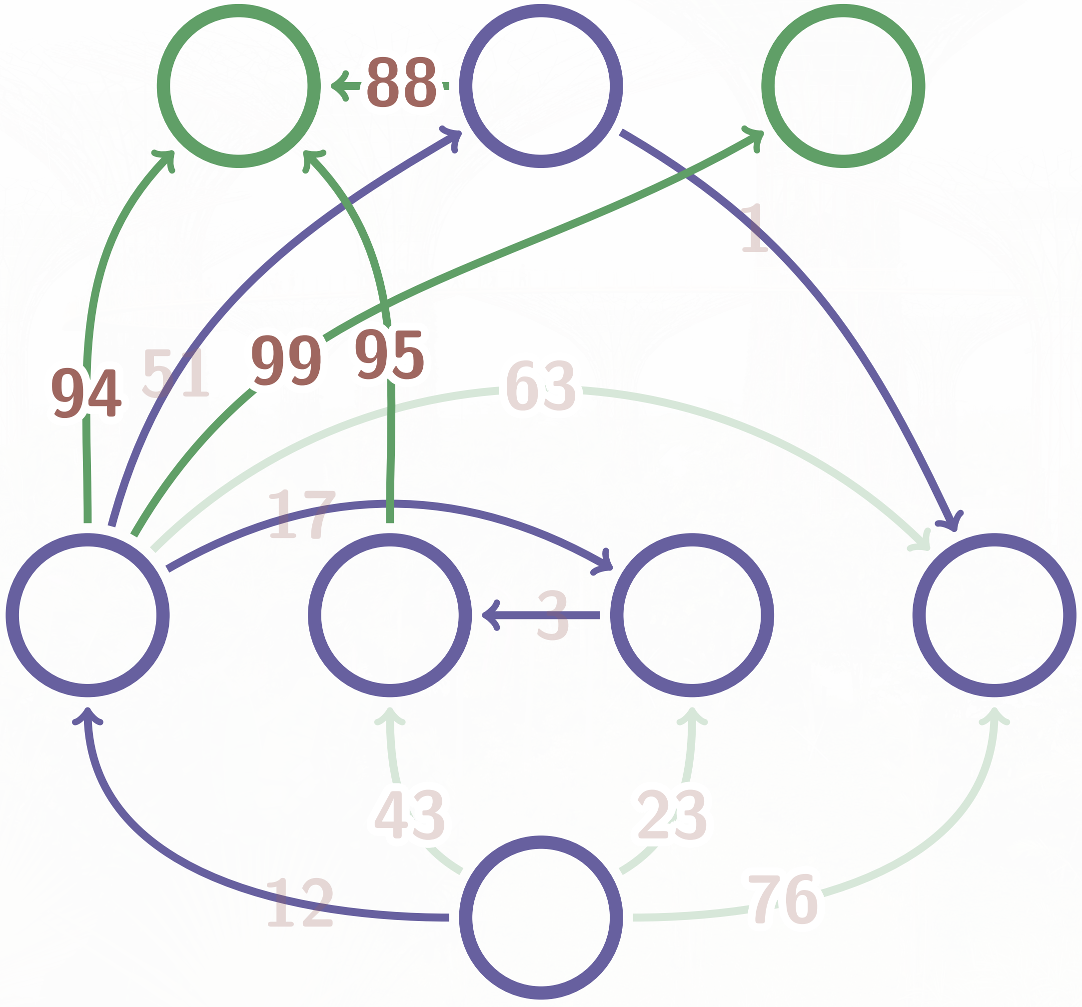

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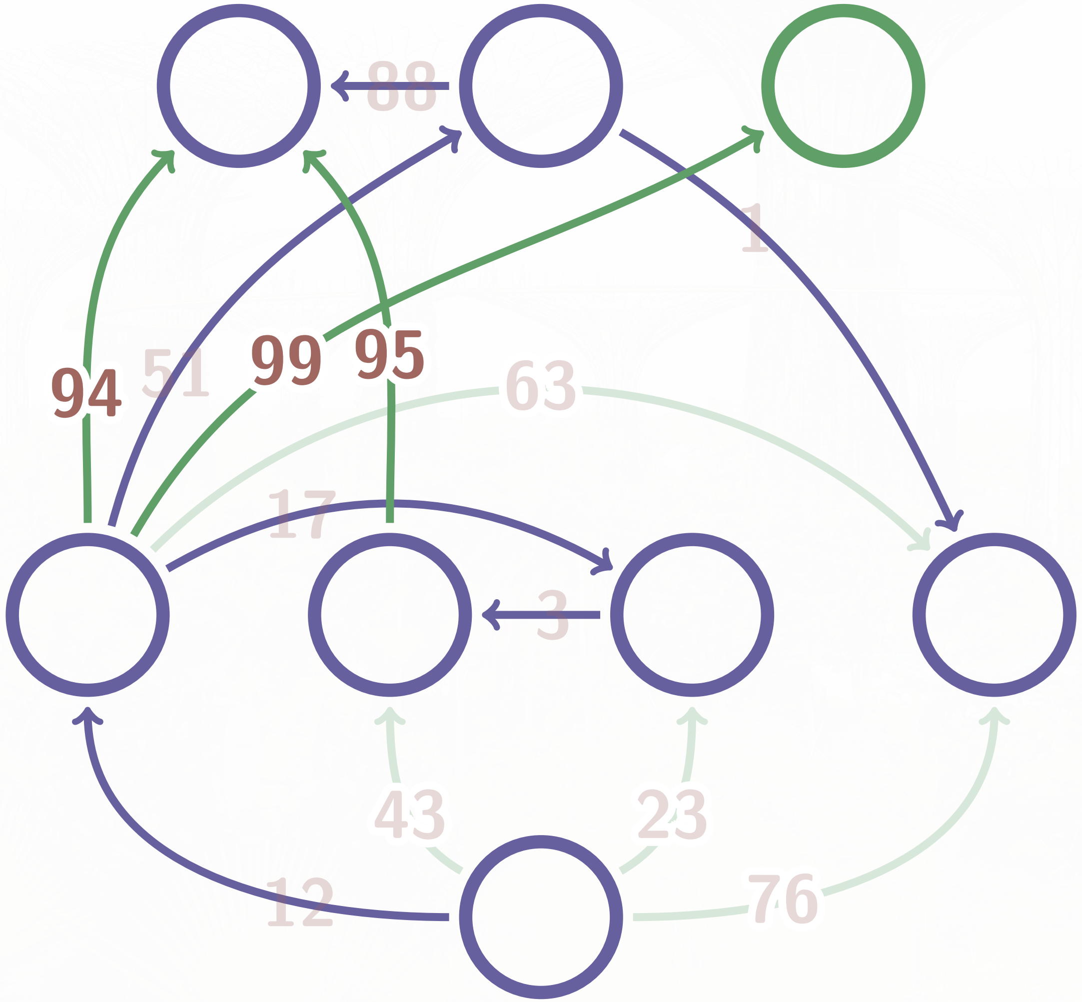

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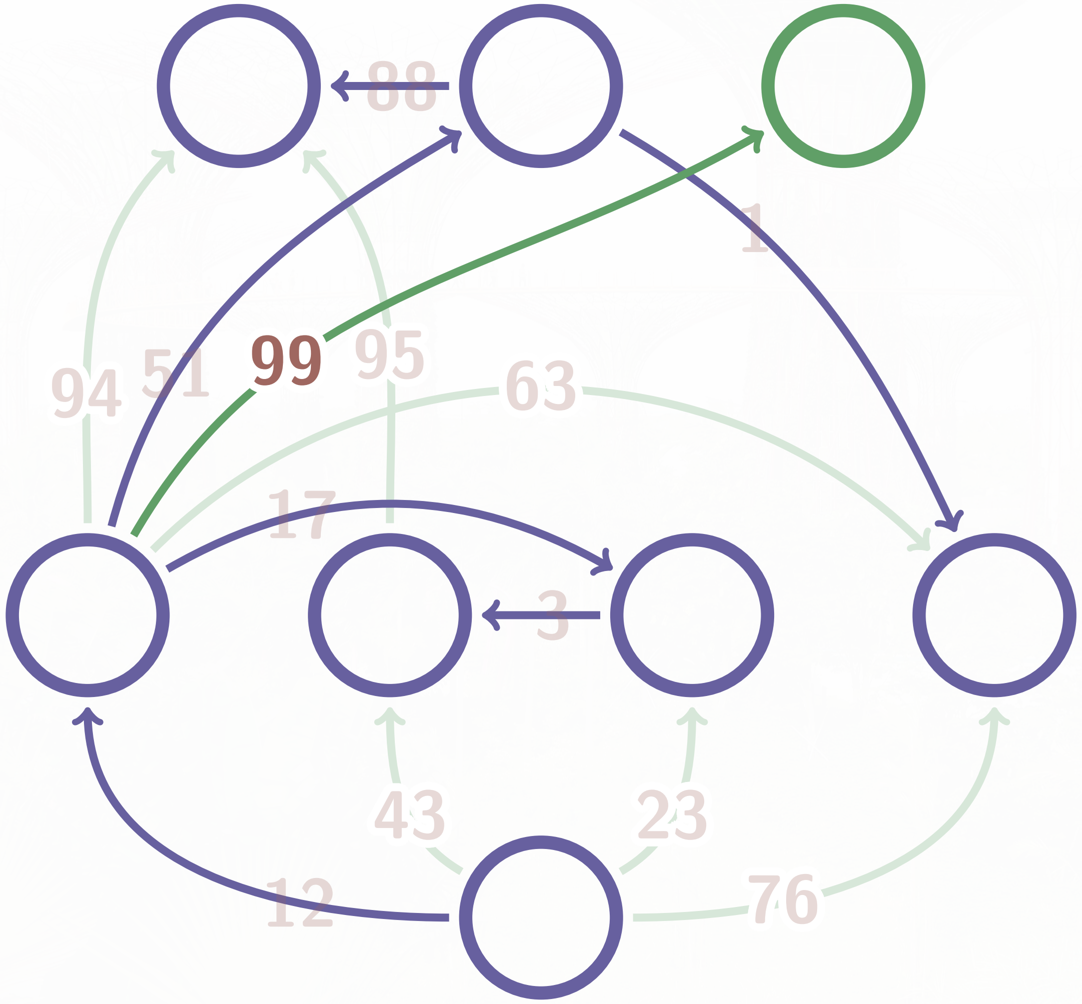

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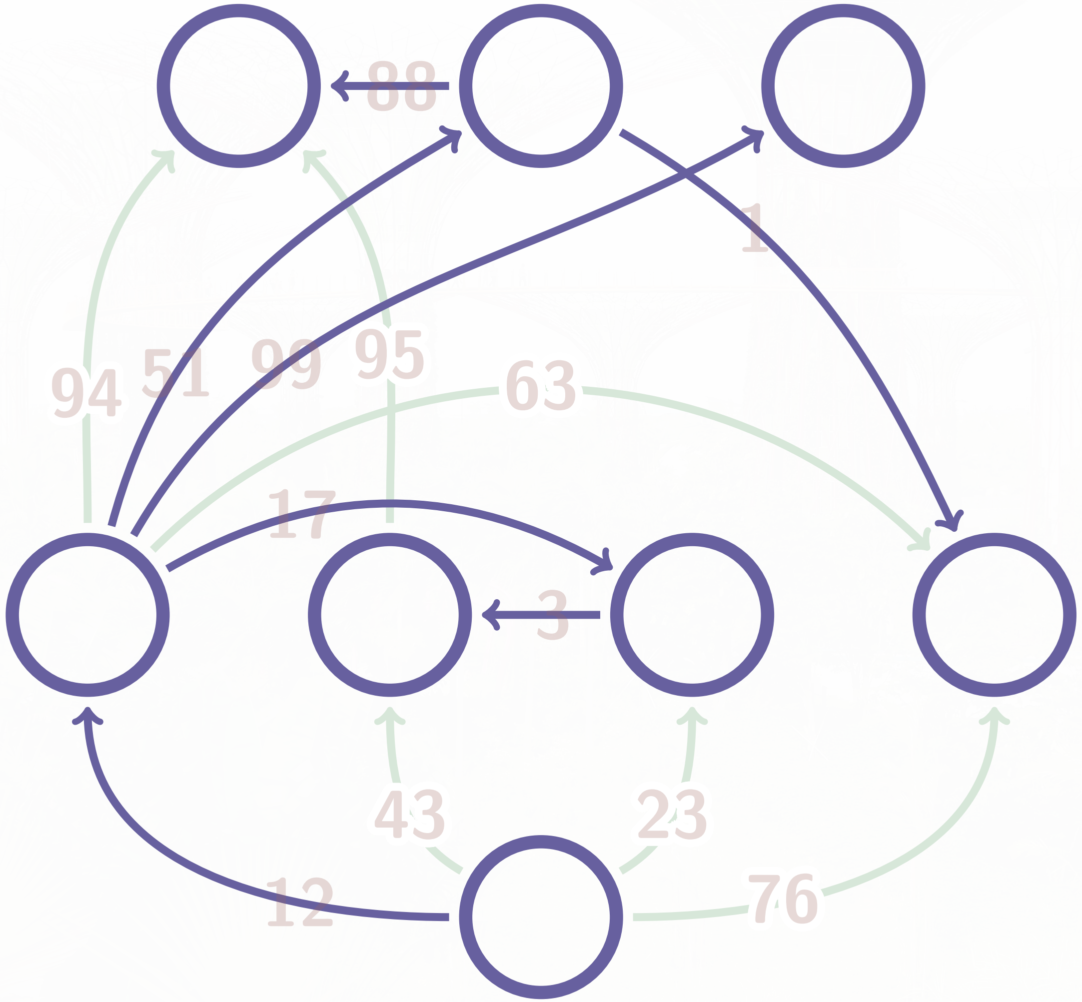

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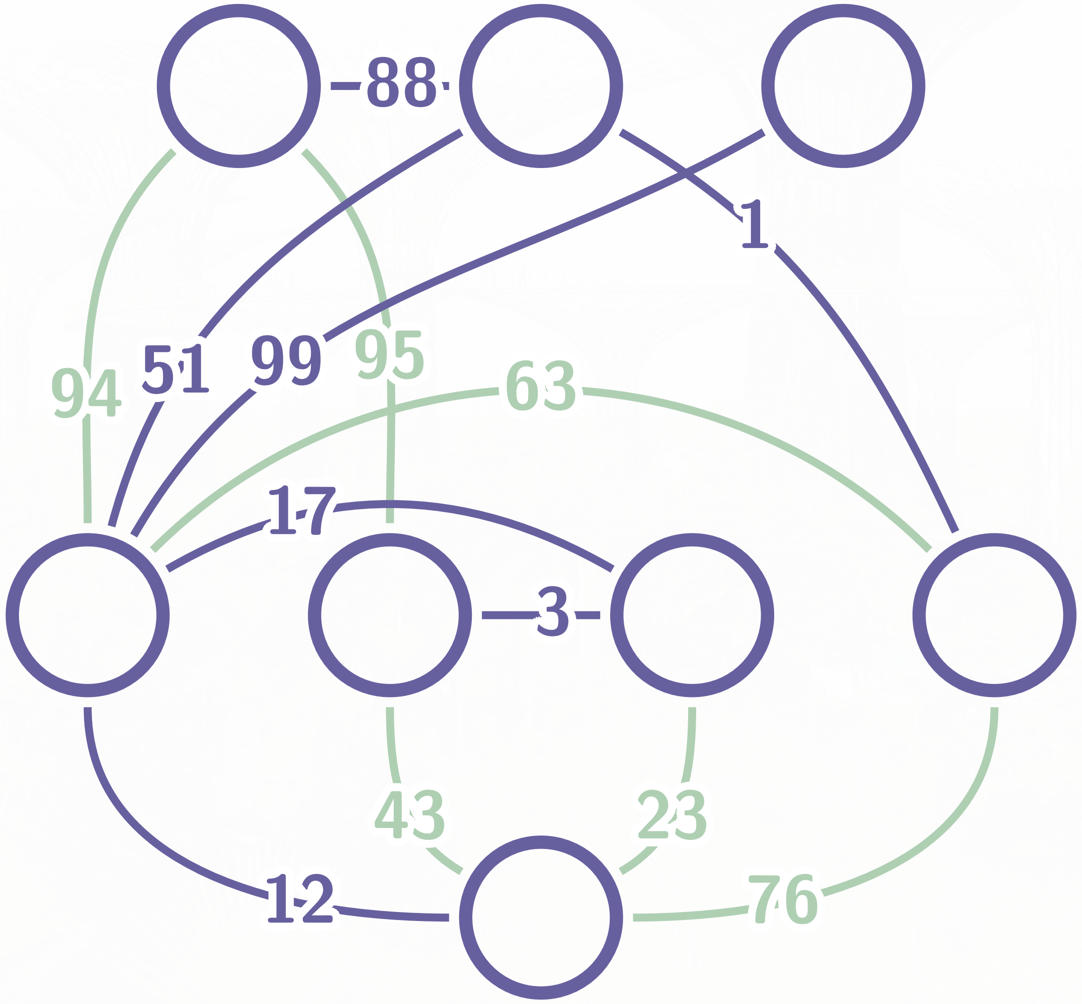

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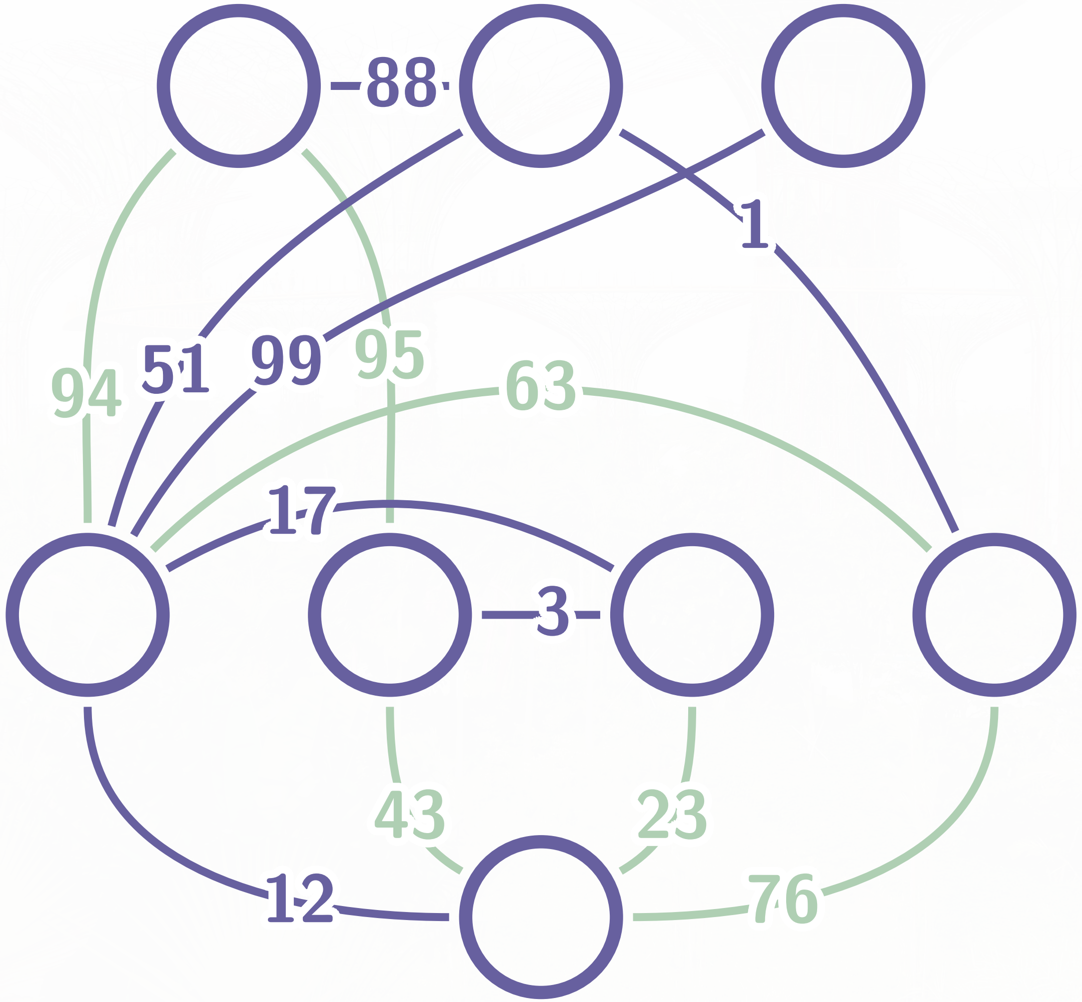

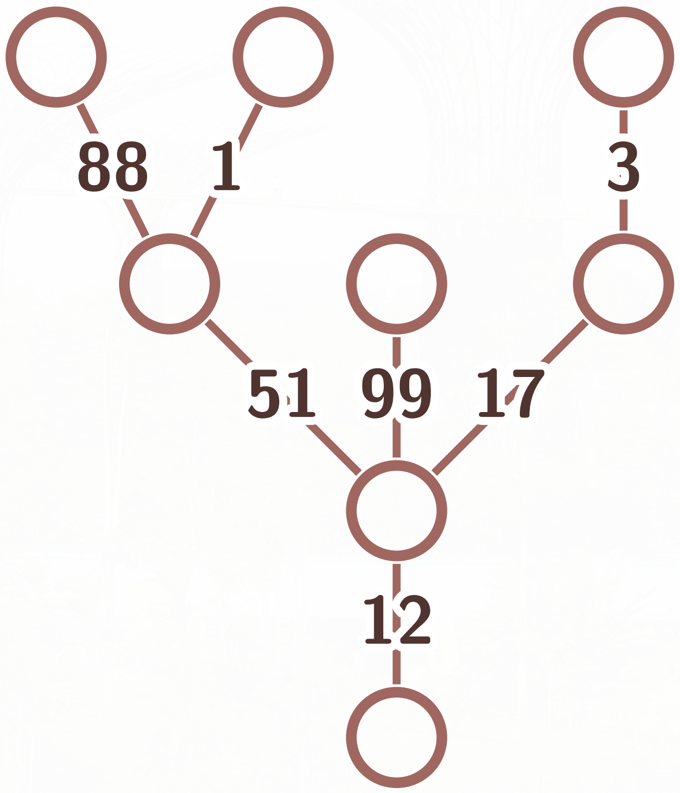

Prim's algorithm and the minimum spanning tree

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The outcome of Prim's algorithm is called the *minimum spanning tree*.

Note: The minimum spanning tree is the unique sub-structure within the graph that connects all the nodes together while minimizing the total sum of its edges.

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→ These concepts of *local* and *global* within a graph may seem opposite but are often more related than expected.

→ In general, global properties are a lot more difficult to study than local properties.

Current research

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Q: Given the local growth, can we deduce global properties of the graph?

Q: Can we provide strong results connecting local and global properties of a graph?

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Universal limits

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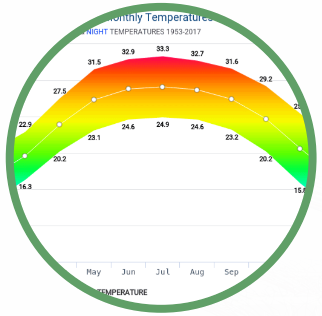

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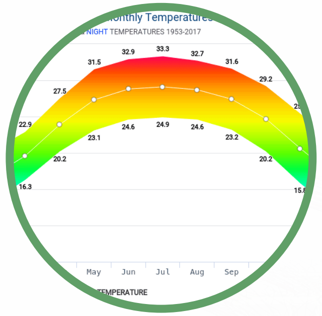

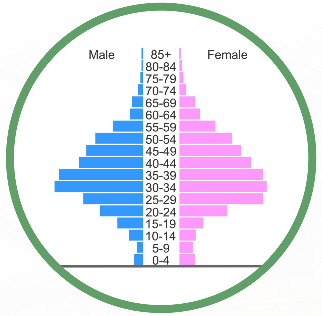

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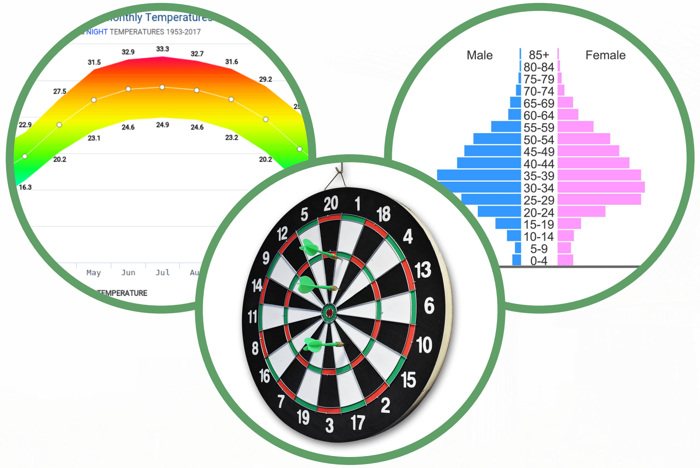

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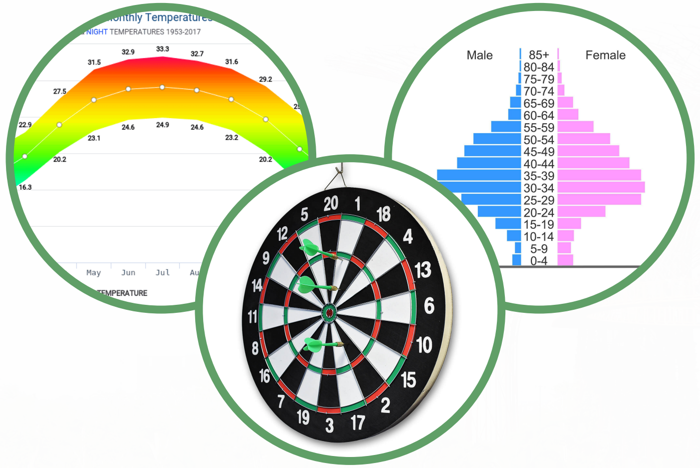

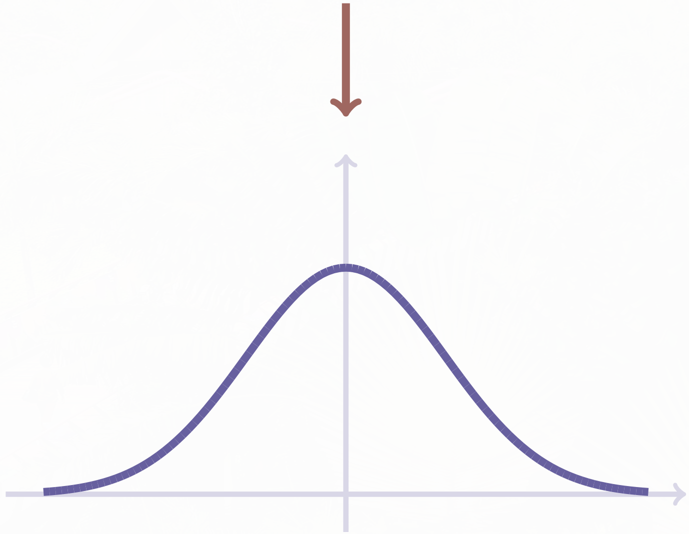

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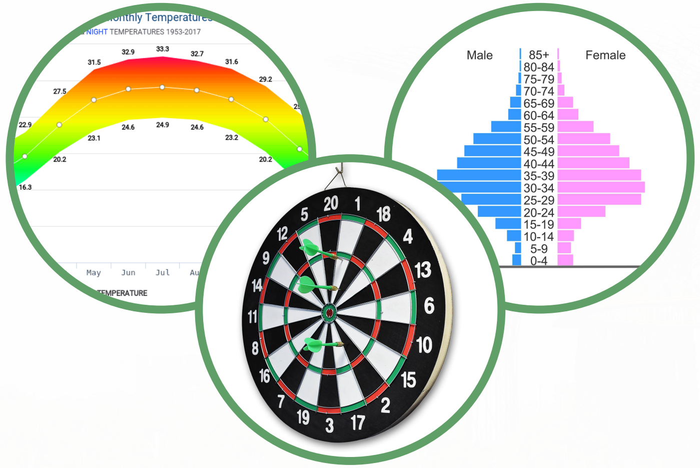

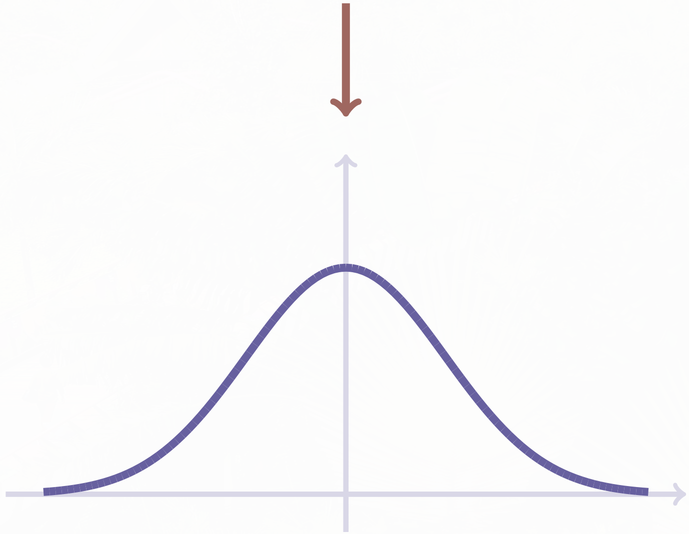

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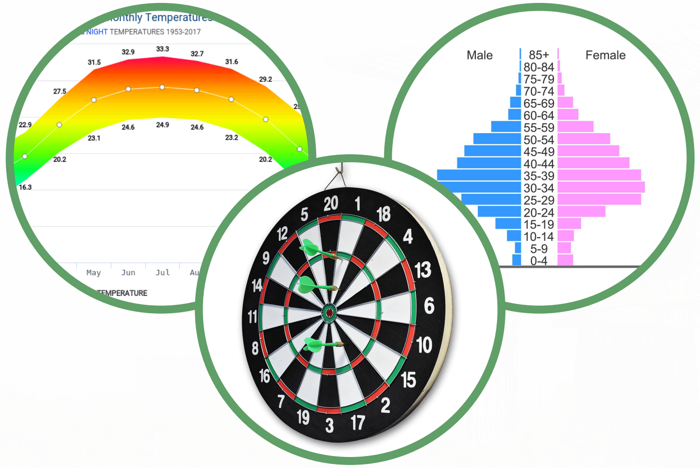

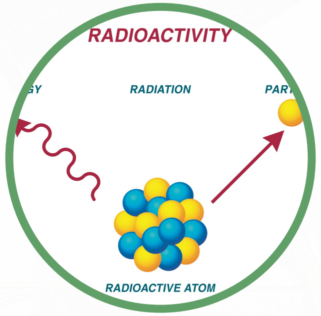

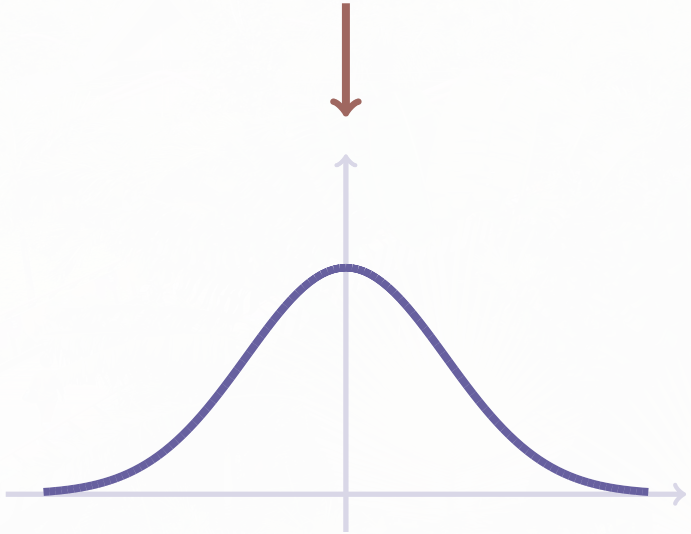

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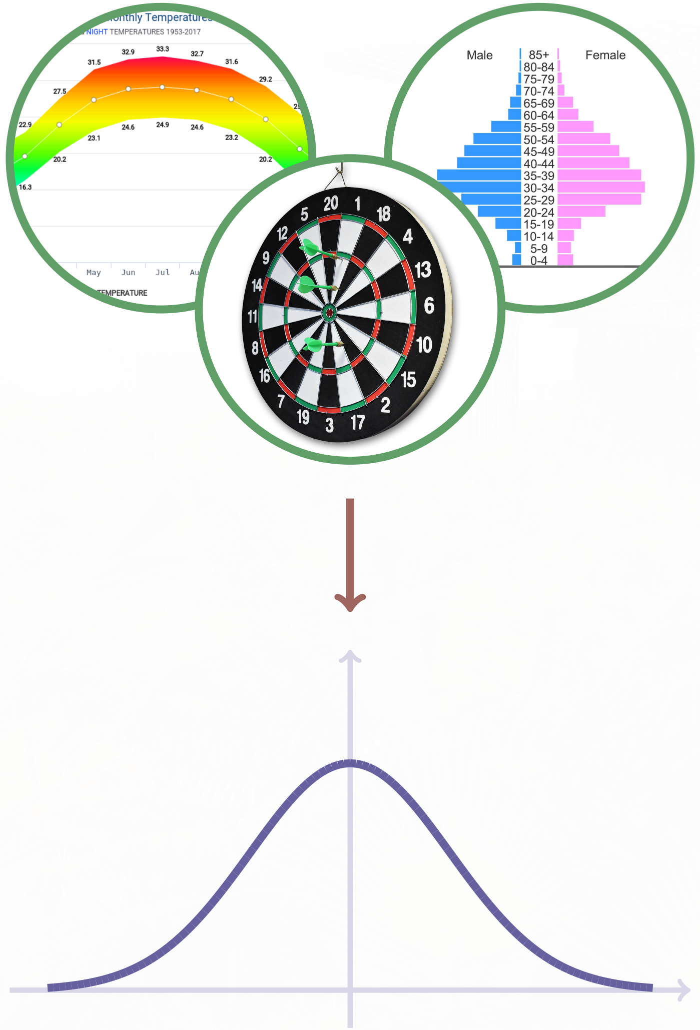

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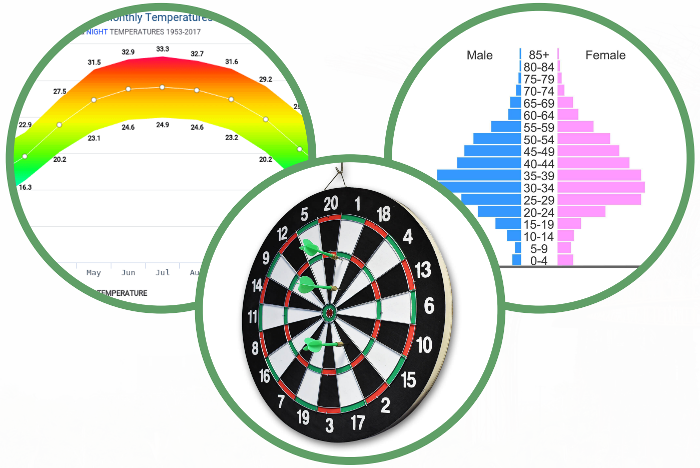

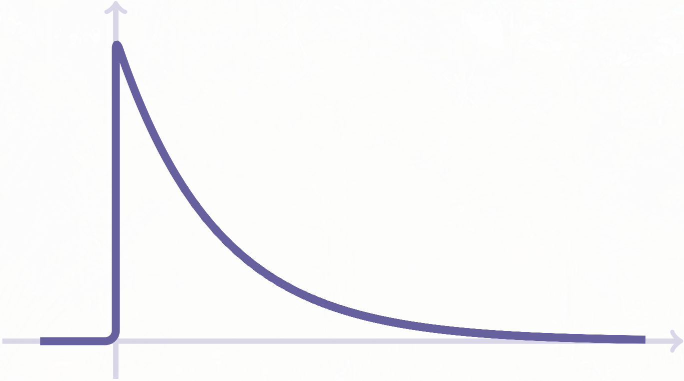

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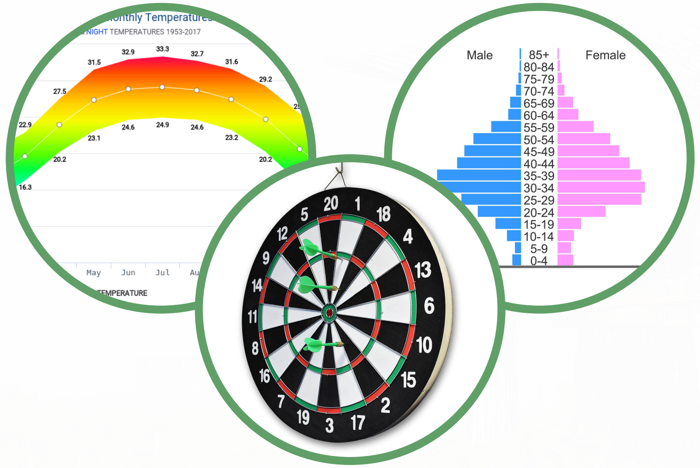

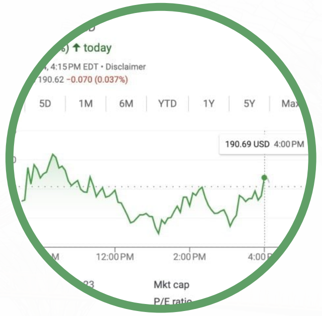

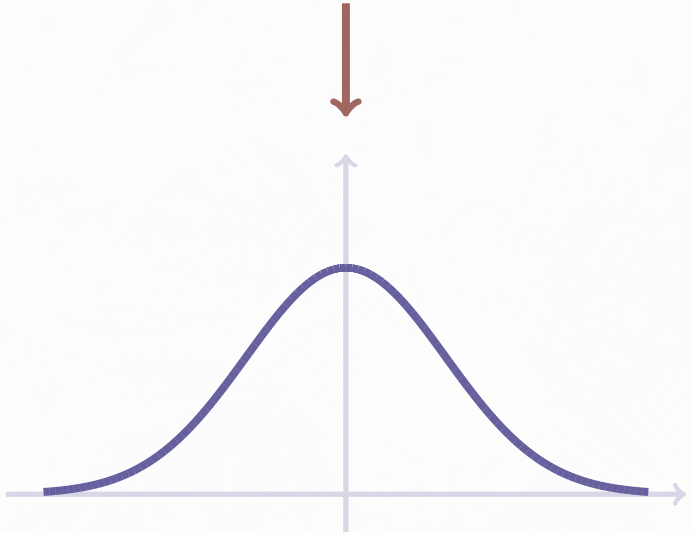

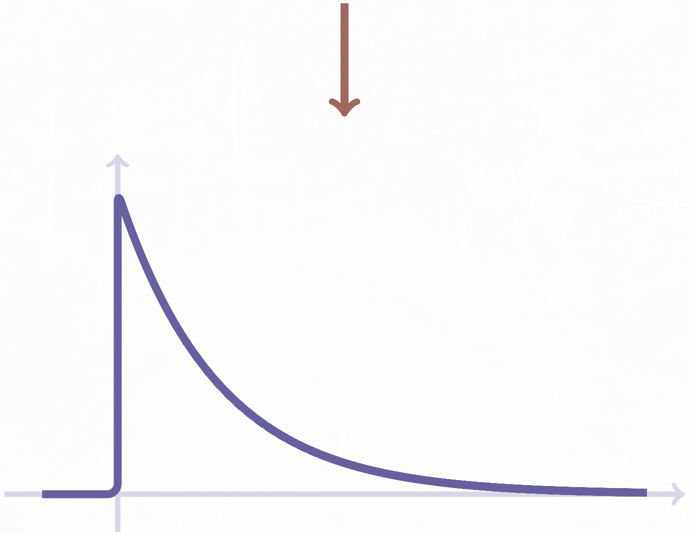

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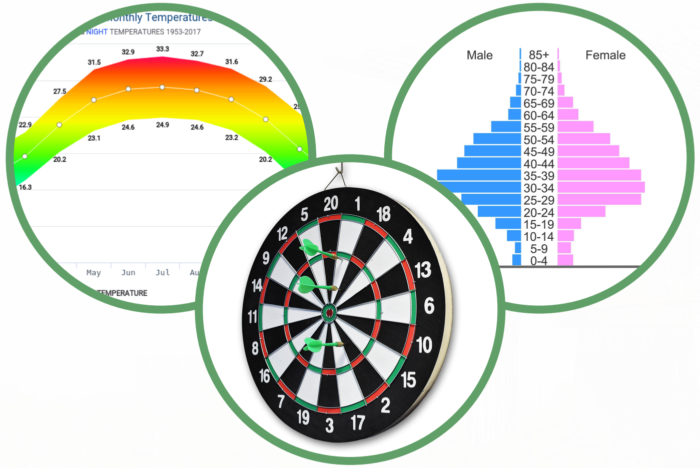

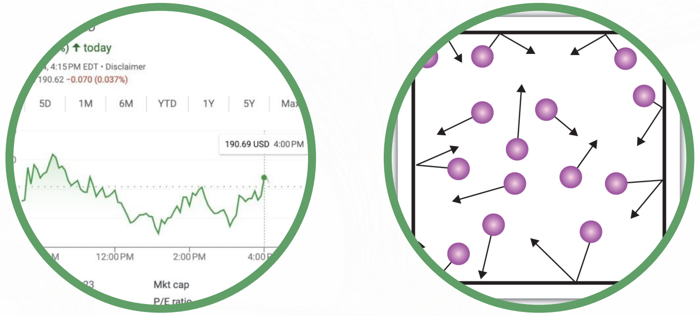

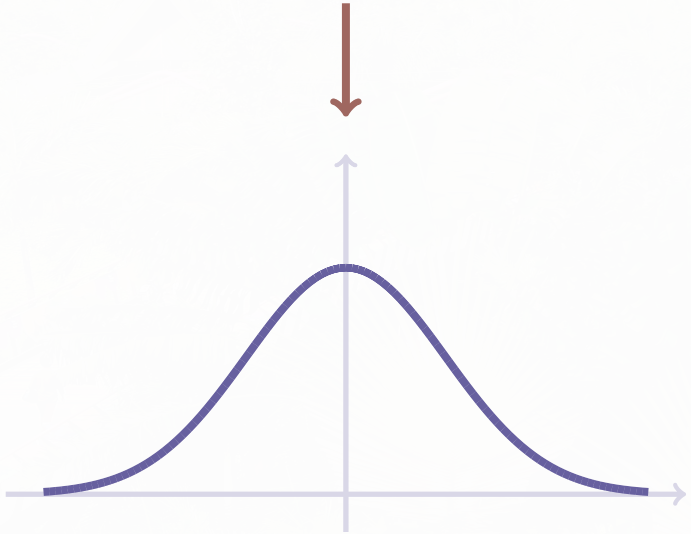

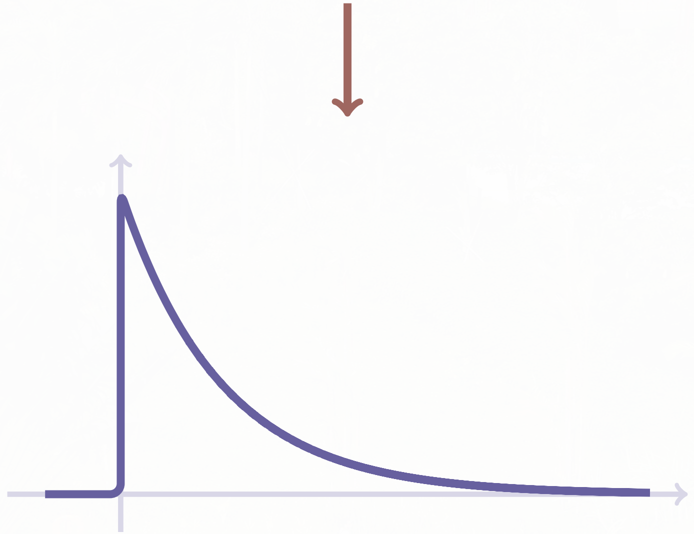

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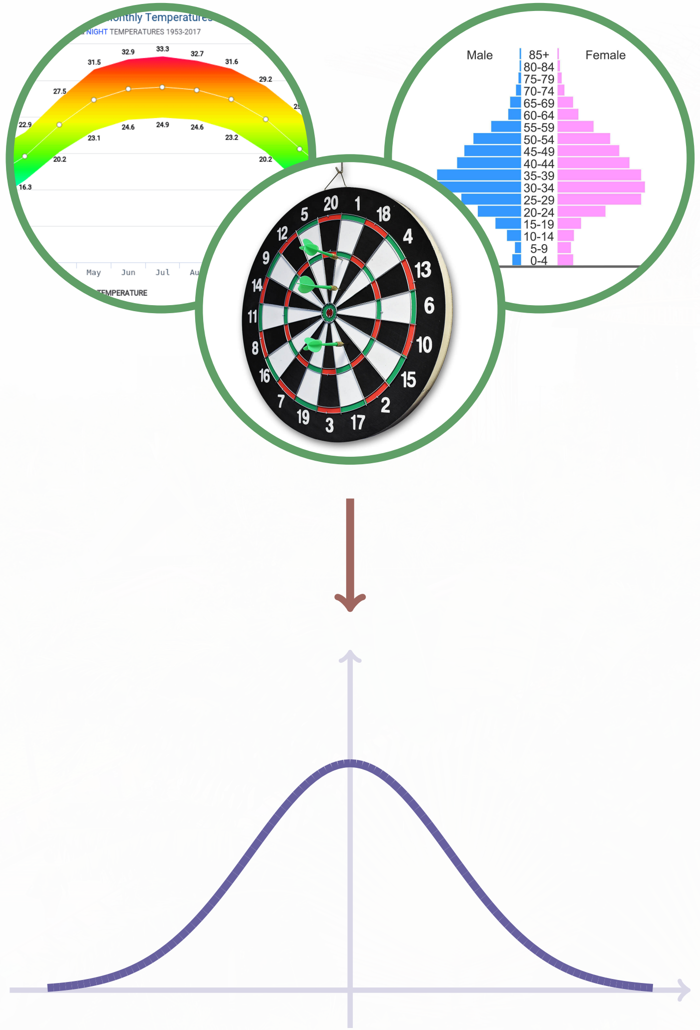

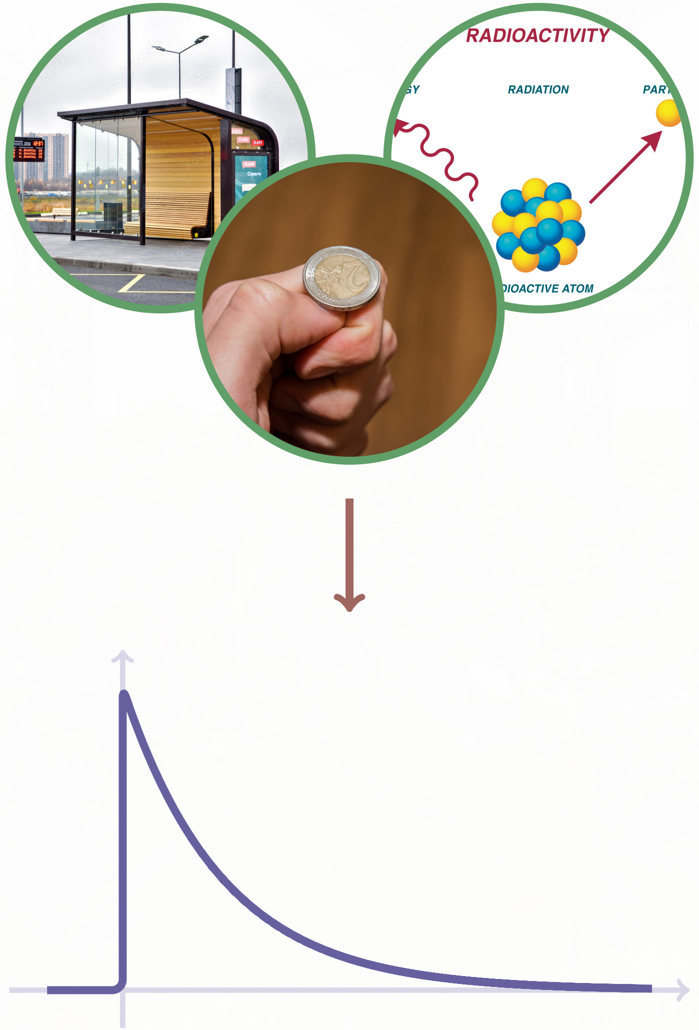

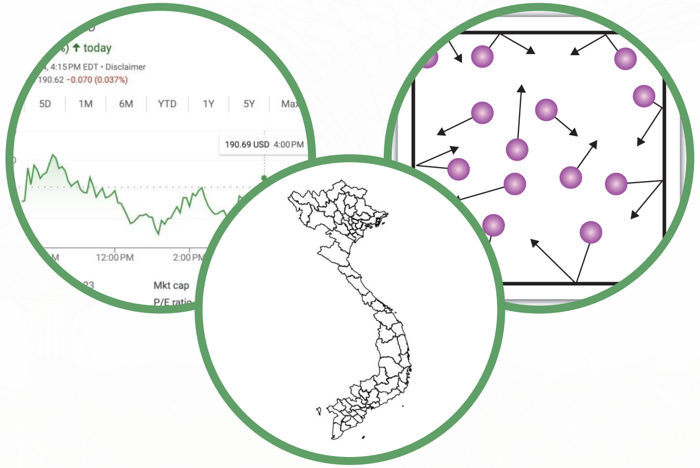

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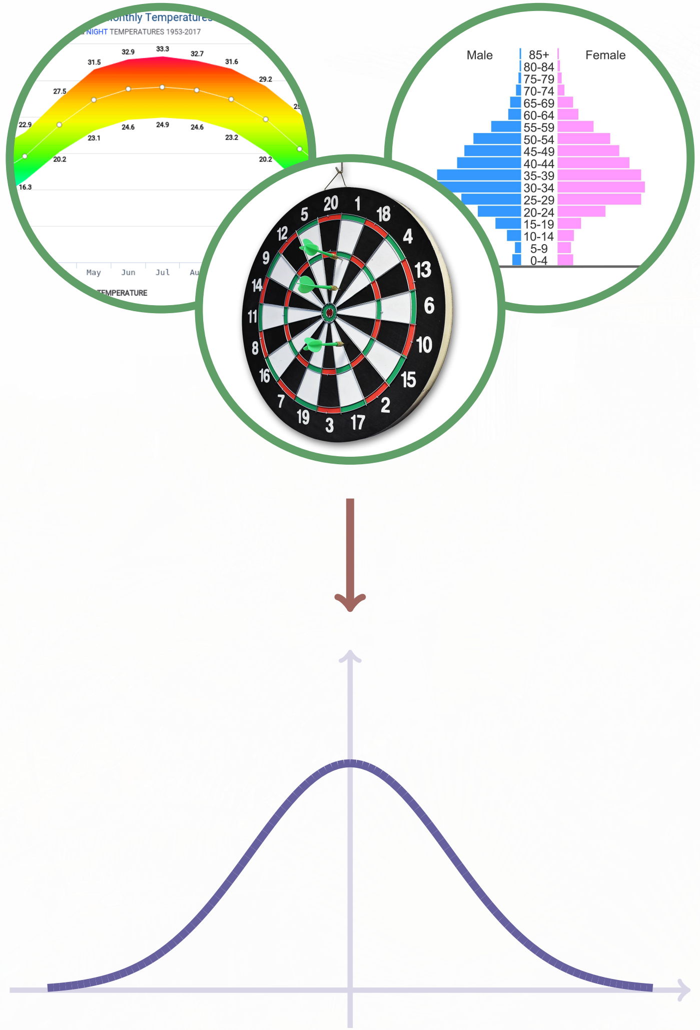

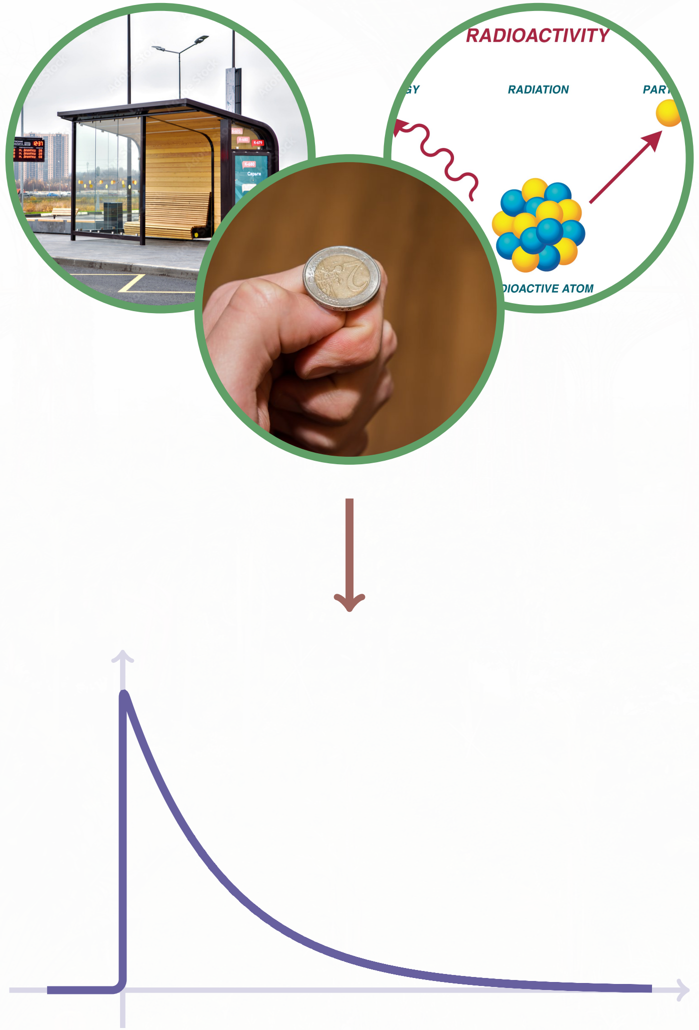

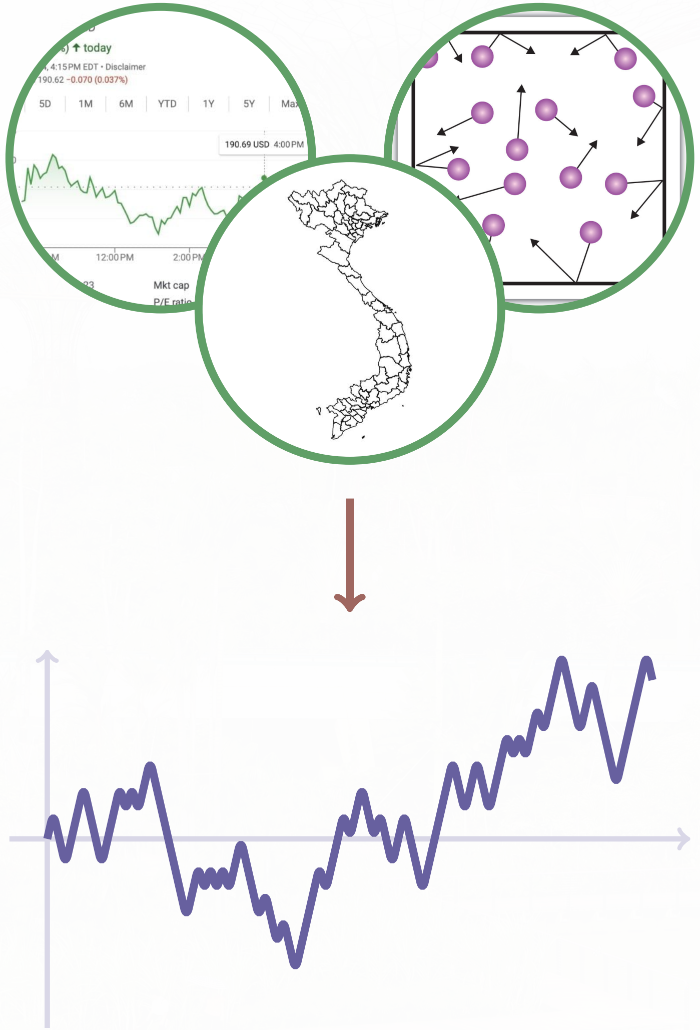

Height-biased trees

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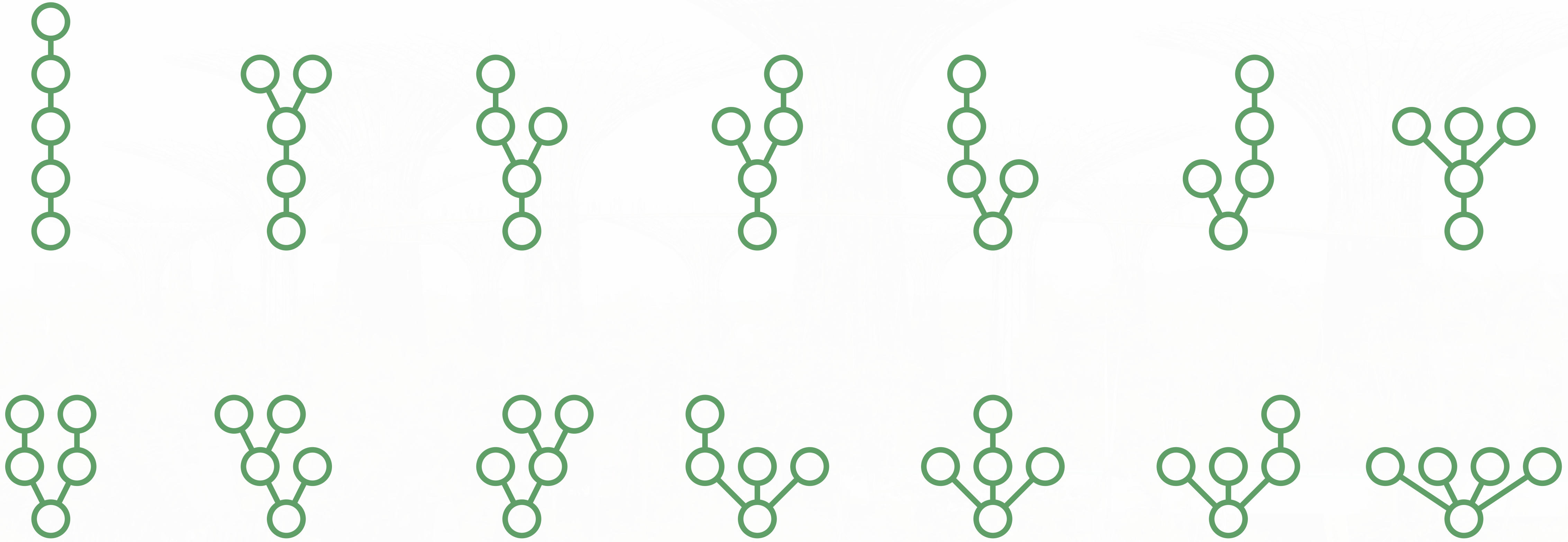

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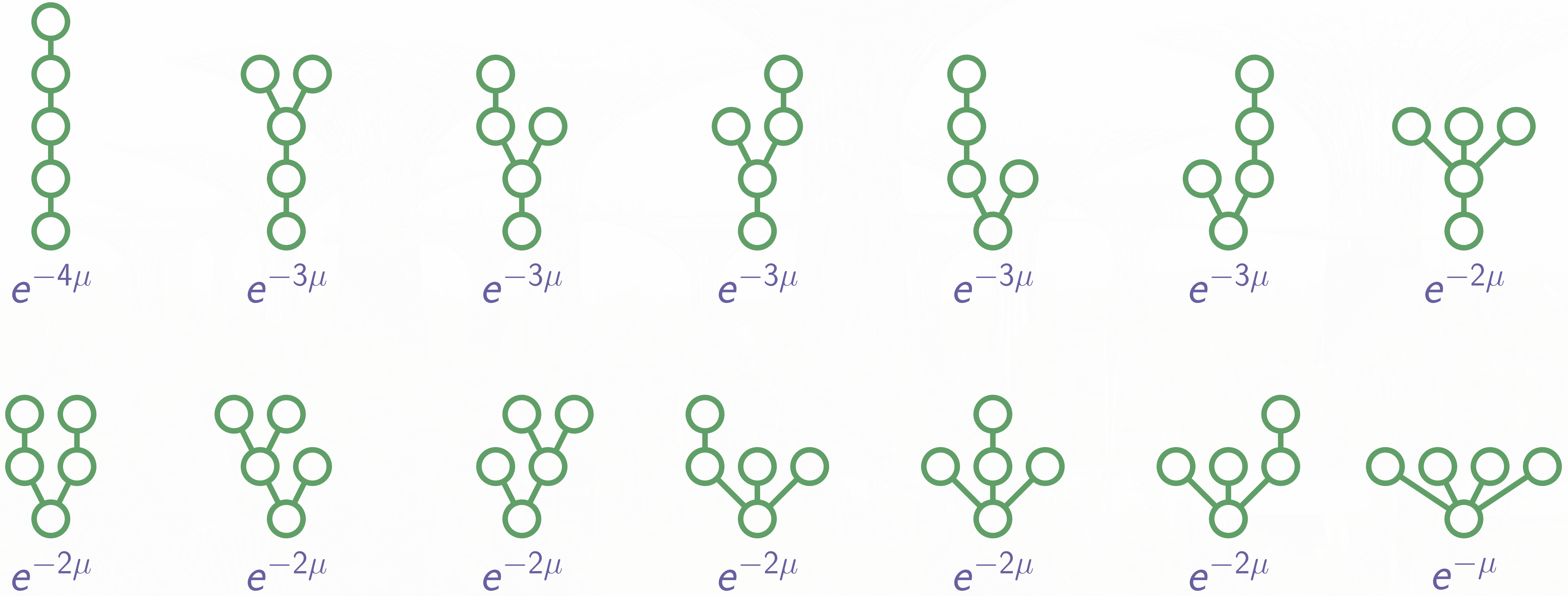

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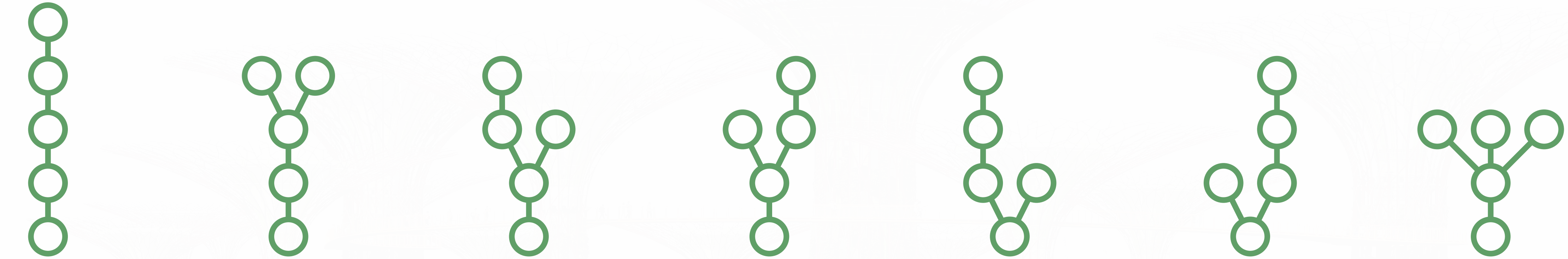

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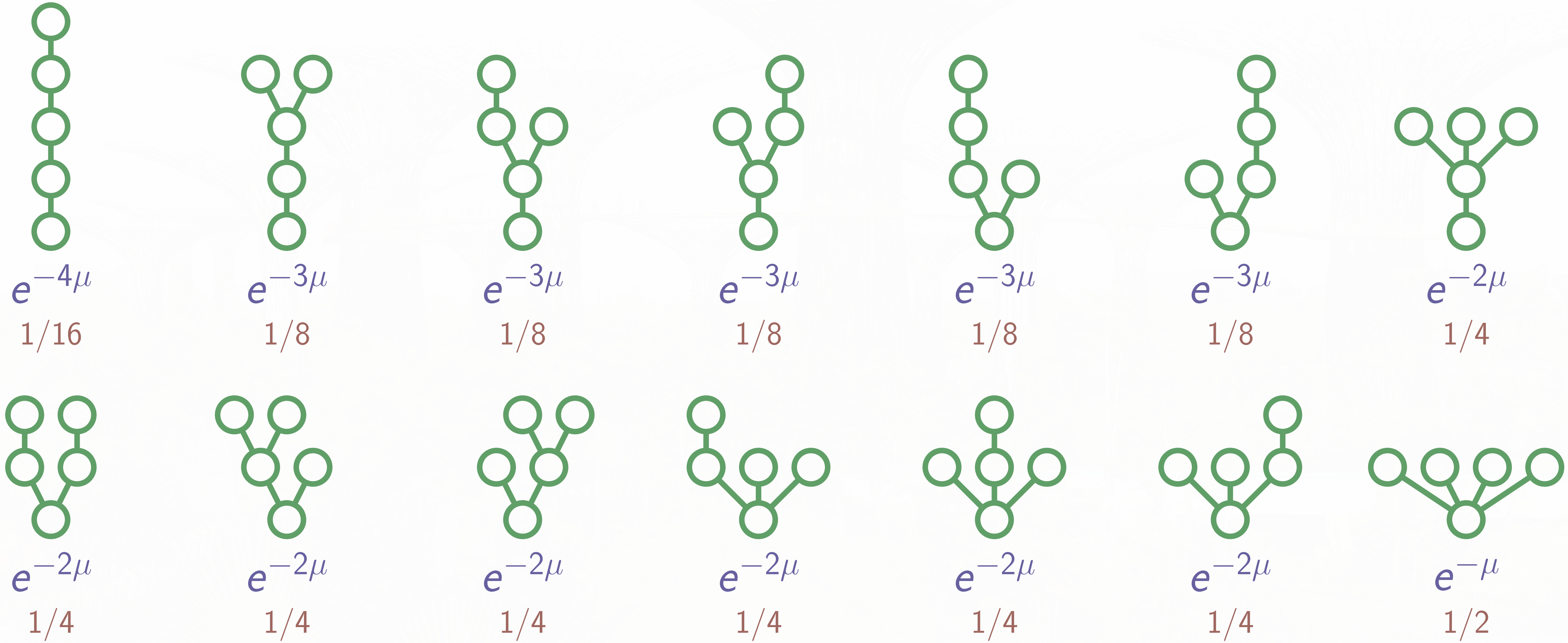

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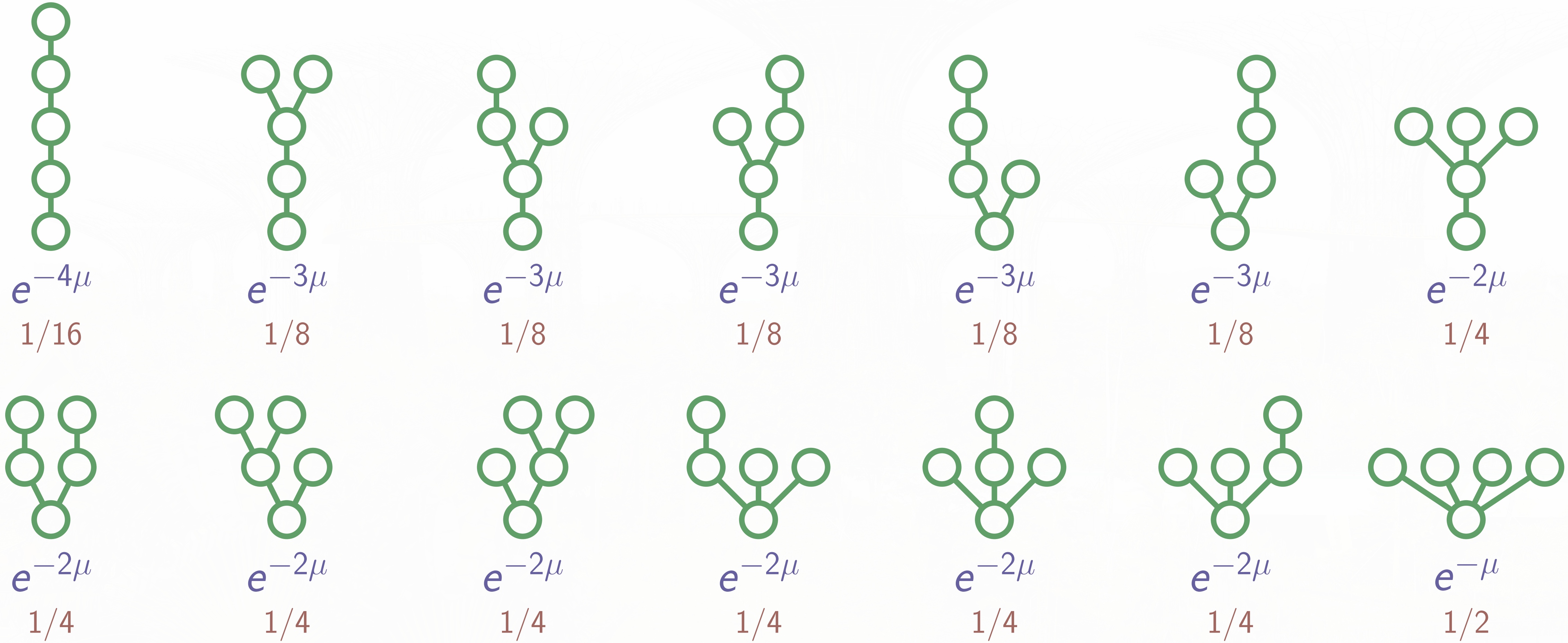

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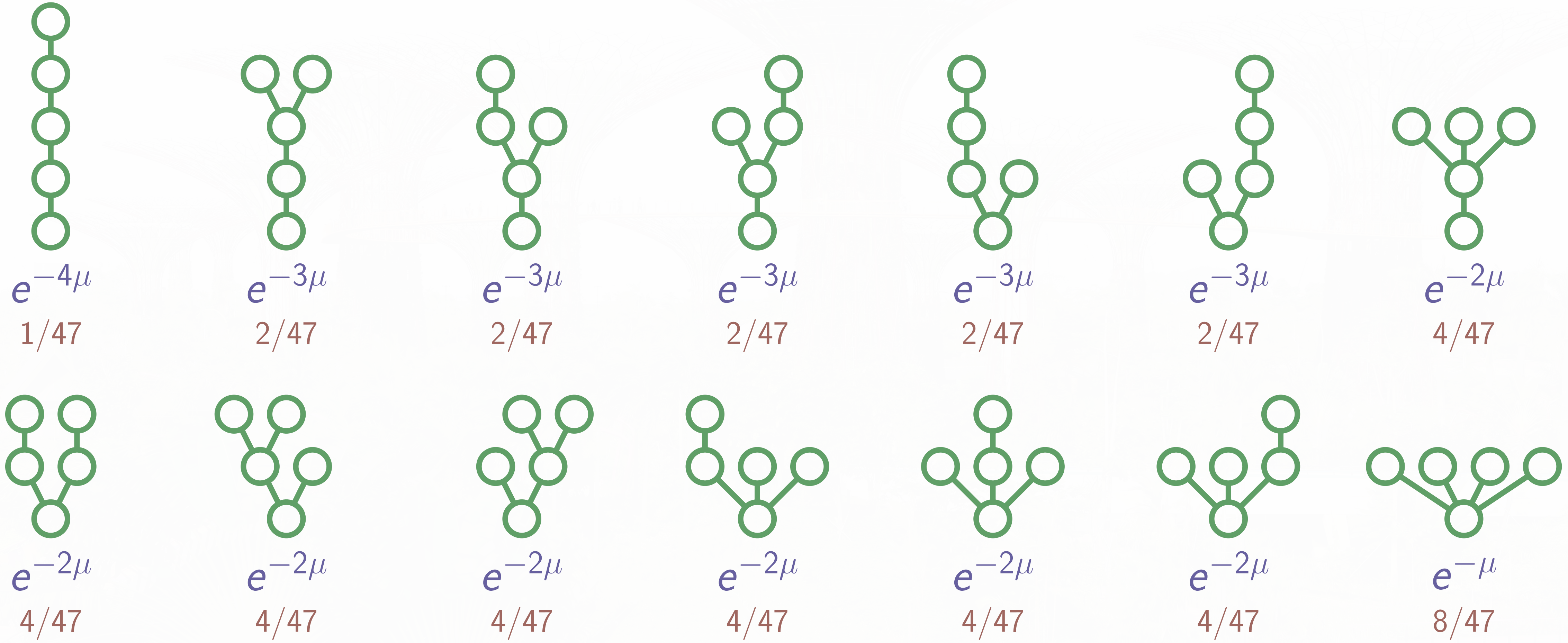

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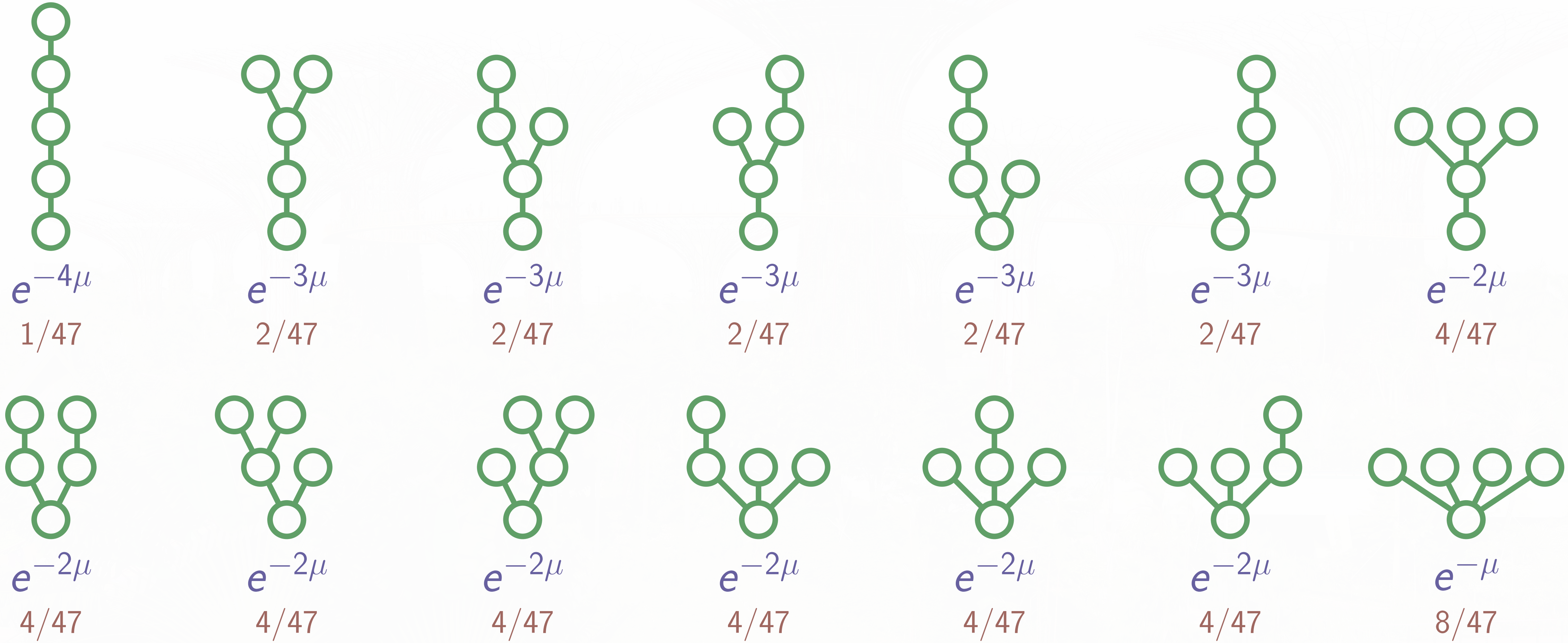

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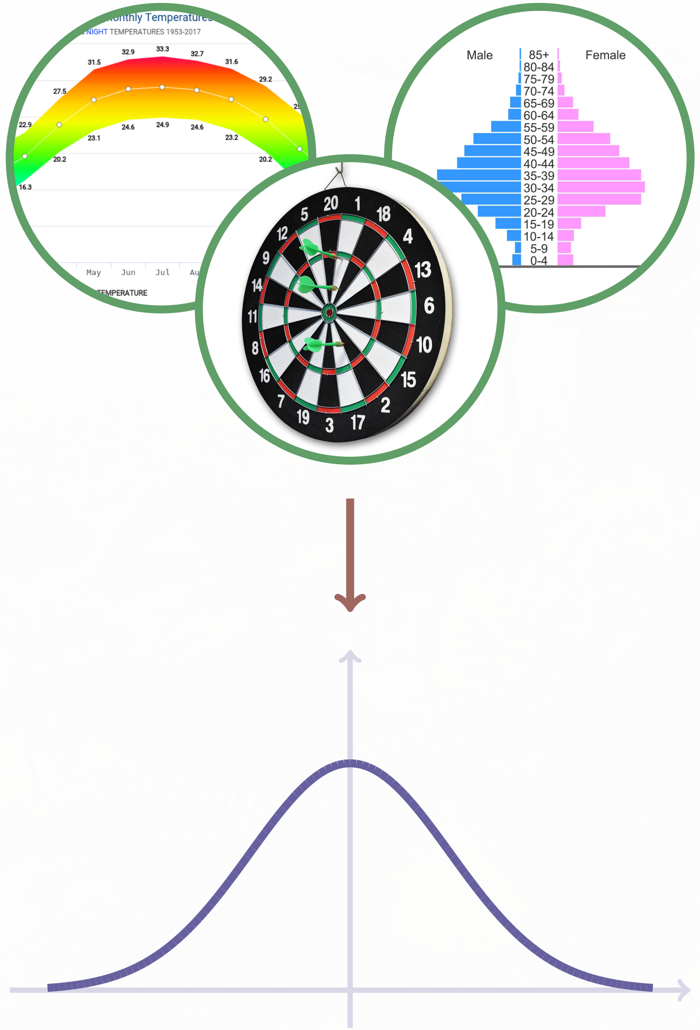

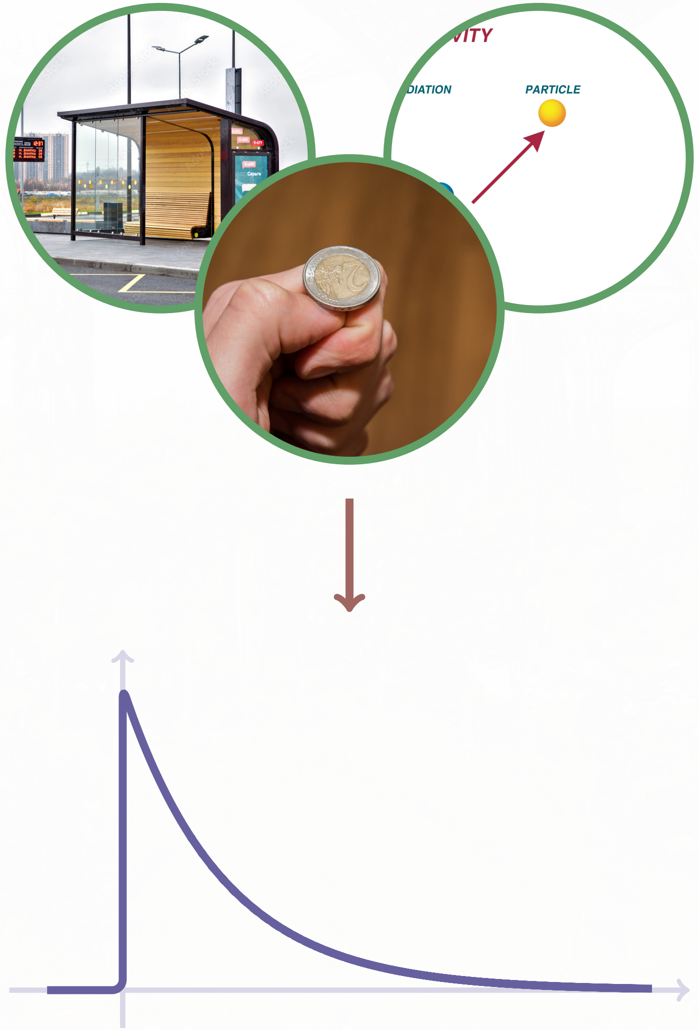

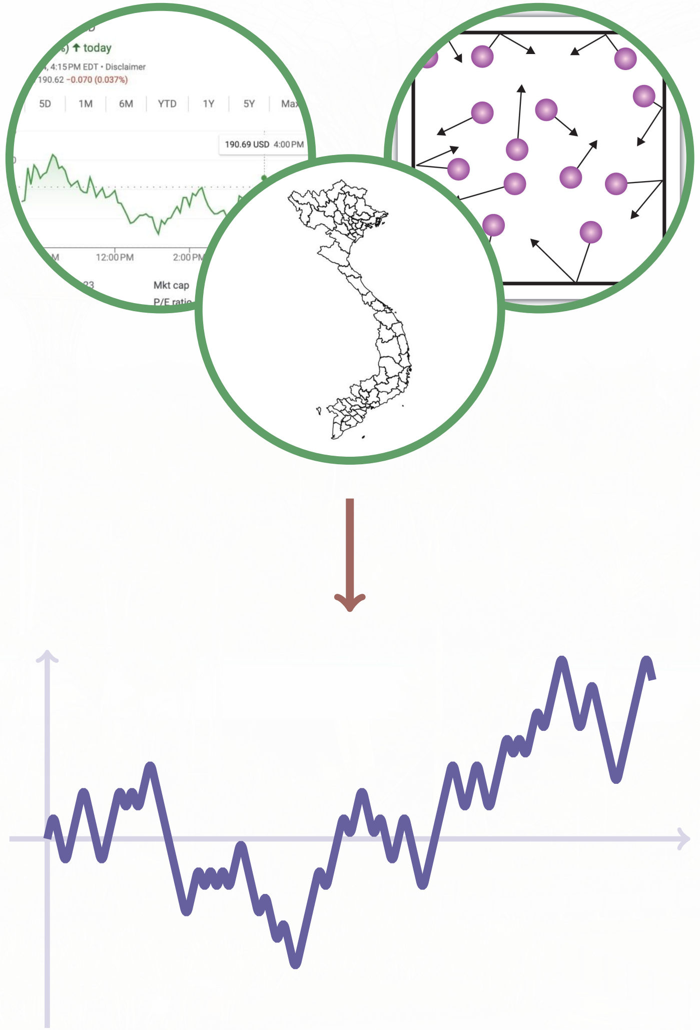

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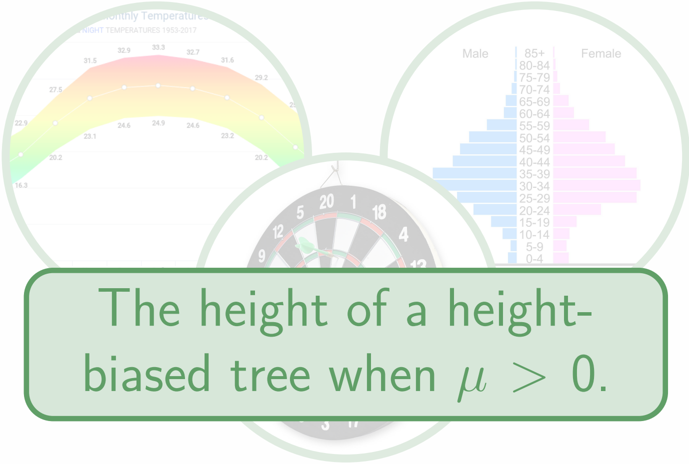
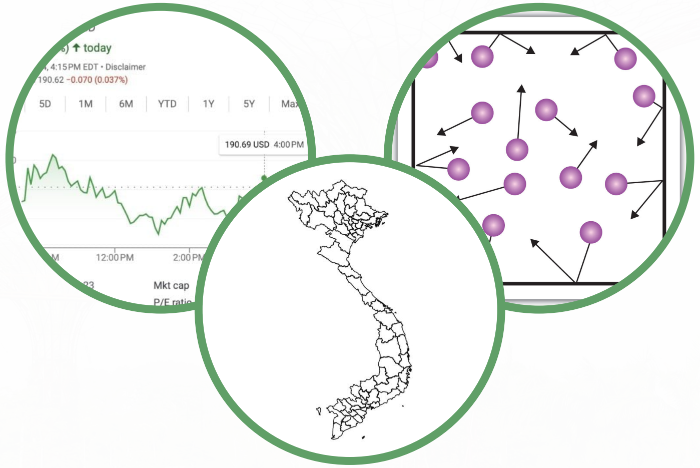
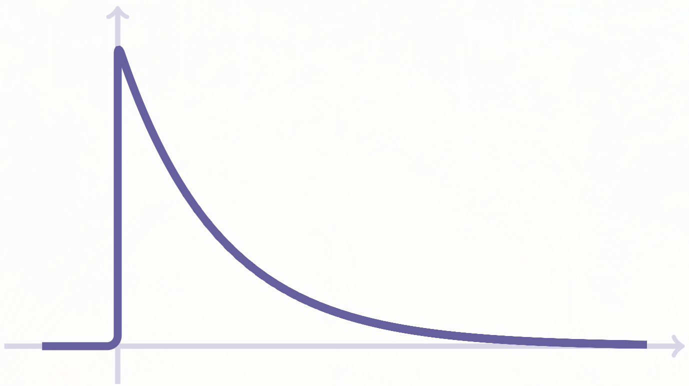
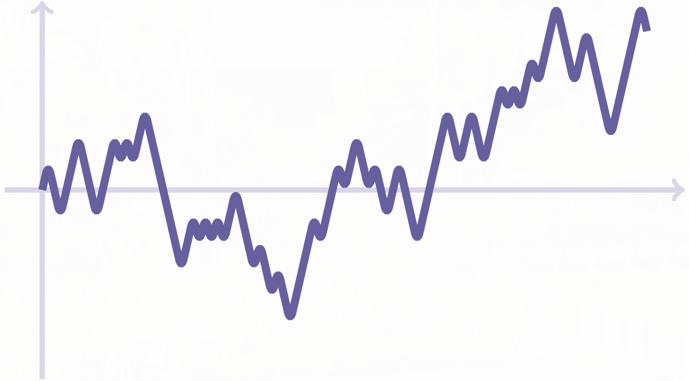


The height of a height-biased tree when $\mu > 0$.







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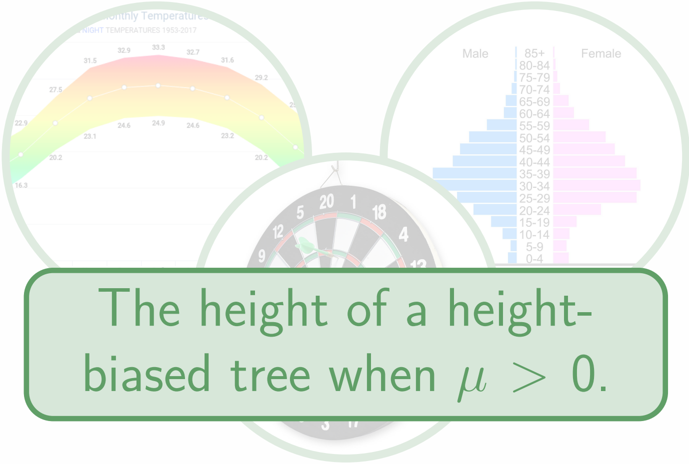

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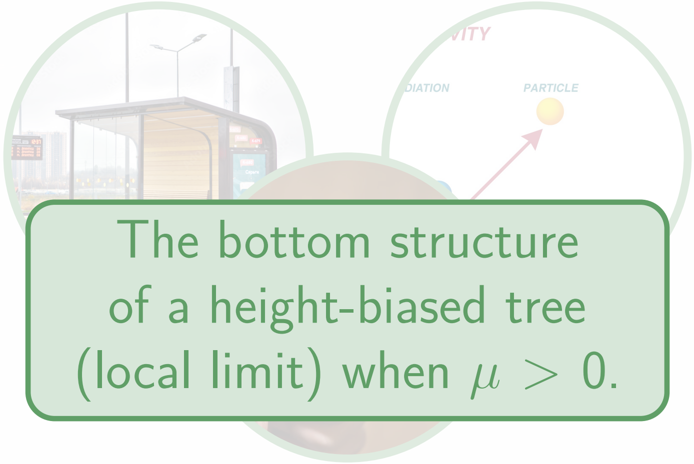
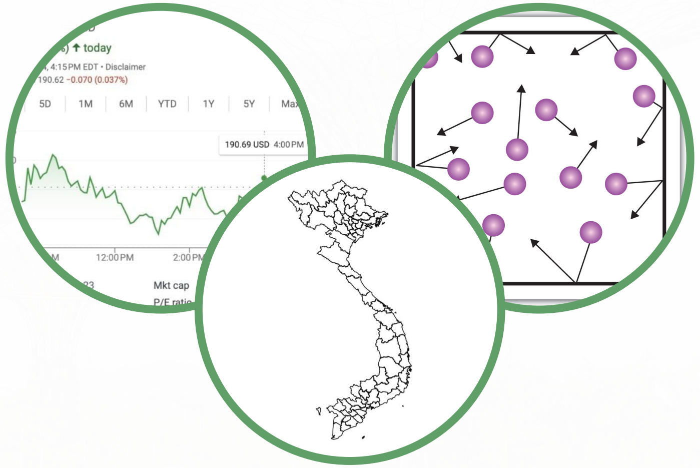
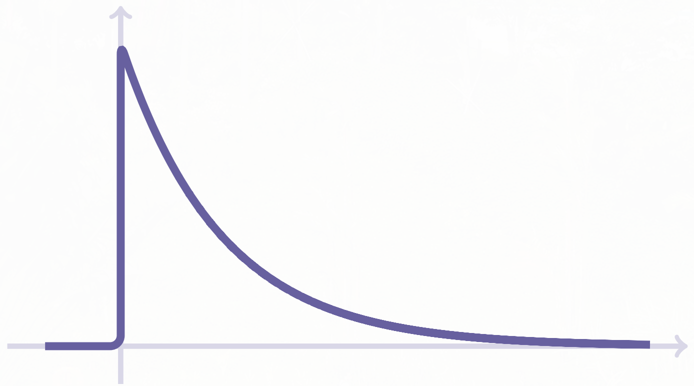
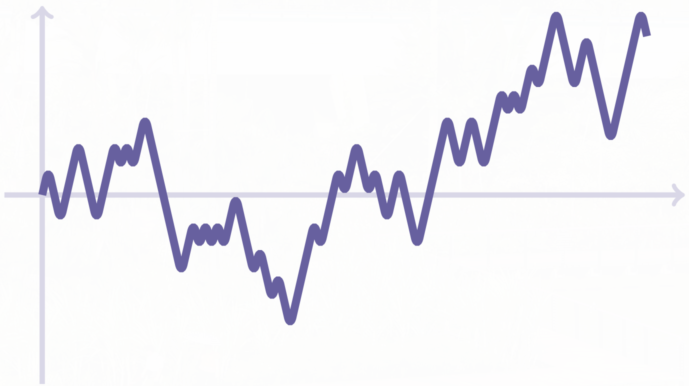


The bottom structure of a height-biased tree (local limit) when $\mu > 0$.







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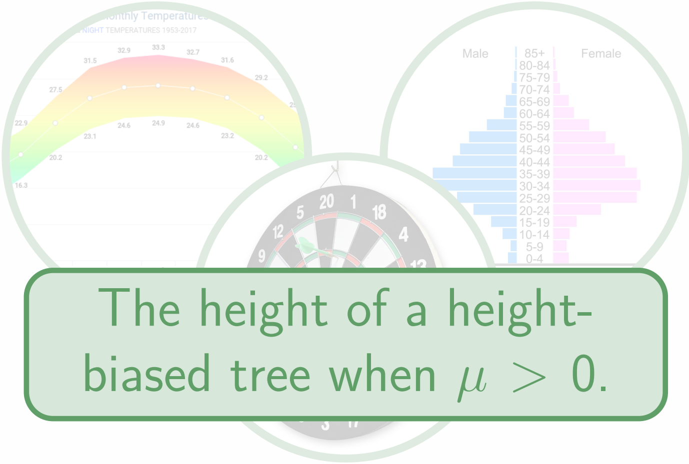

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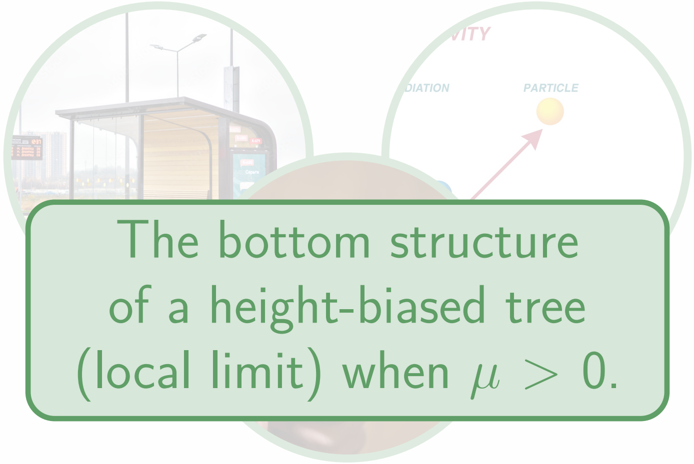

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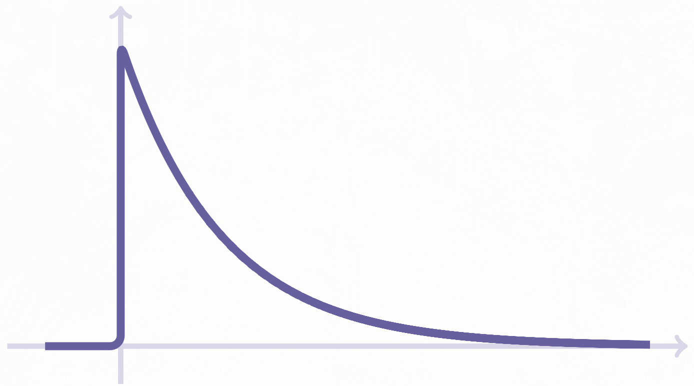
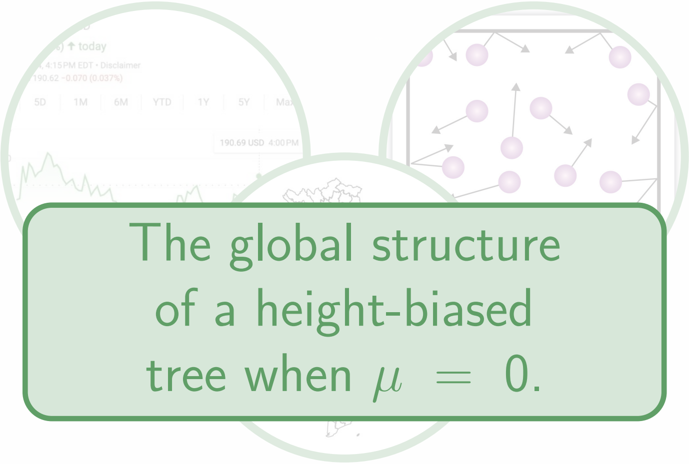
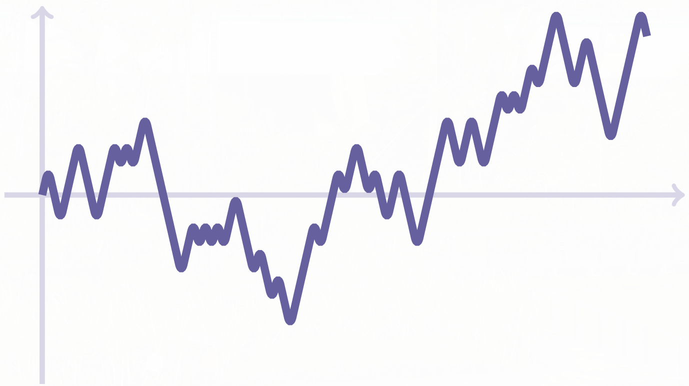




The global structure of a height-biased tree when $\mu = 0$.



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Q: Why does this model show such universality results?

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Thank you! Thank you! Thank you!

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